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On Certain Subclass of Harmonic Univalent Functions

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Abstract - In this paper we investigate a class of harmonic univalent functions obtaining its coefficient inequality, growth and distortion theorems and convolution properties.

Keywords- *Univalent Functions, Harmonic Functions*

I. INTRODUCTION

A continuous complex valued function $f = u + iv$ defined in a simply connected domain D is said to be harmonic in D if both u and v are real harmonic in D. the harmonic function has a unique representation $f = h + \overline{g}$, i.e. there exists analytic functions H and G such that

$$f = \frac{H+\overline{H}}{2} + \frac{G-\overline{G}}{2} = \left(\frac{H+G}{2}\right) + \left(\overline{\frac{H-G}{2}}\right) = h + \overline{g}$$

where h and g are analytic and co analytic part of f respectively.The Jacobian of $f = h + \overline{g}$ is given by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$. The mapping $z \to f(z)$ is orientation preserving and locally 1-1 in D if and only if $J_f(z) > 0$ in D (see Lewy [6] and Clunie and Shiel small [2]). Let h denote the family of normalized functions $f = h + \overline{g}$ that are harmonic, orientation preserving and univalent in the open unit disk $\Delta = \{z : |z| < 1\}$ see [3, 7] where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \ |b_1| < 1 \tag{1}$$

here

$$z' = \frac{\partial}{\partial\theta}(z = re^{i\theta}), f'(z) = \frac{\partial}{\partial\theta}(f(z)) = \frac{\partial}{\partial\theta} f(re^{i\theta}),$$

$0 \le r < 1, \ \theta \in \mathbb{R}.$

Several researchers have defined and studied new subclasses of harmonic univalent functions see [1, 4, 5, 8]. In this paper we introduce a subclass of harmonic univalent functions and obtain the coefficient inequality, growth estimate, distortion theorem and convolution properties for the functions in this class.

For $0 \le \beta < 1$, we consider the subclass $L_H(\beta)$ of harmonic univalent functions $f = h + \overline{g}$ satisfying the condition

$$\operatorname{Re}\left\{(1+e^{i\alpha})\left(\frac{z^2 f''(z) + zf'(z)}{f(z)}\right) - e^{i\alpha}\right\} \ge \beta \tag{2}$$

Further let $L_{\overline{H}}(\beta)$ denote the subclass of $L_H(\beta)$ consisting of functions $f = h + \overline{g}$ such that

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad g(z) = \sum_{n=1}^{\infty} |b_n| z^n \tag{3}$$

II. MAIN RESULTS

Theorem 1 Let $f = h + \overline{g}$ be such that h and g are given by (1), then f is harmonic univalent in Δ and $f \in L_{\overline{H}}(\beta)$, if

$$\sum_{n=1}^{\infty}\left[\frac{2n^2-1-\beta}{1-\beta}|a_n| + \frac{2n^2+1+\beta}{1-\beta}|b_n|\right] \le 2 \tag{4}$$

Where $a_1 = 1, \ 0 \le \beta < 1$

Proof f is locally univalent and orientation preserving in Δ, if $|h'(z)|^2 - |g'(z)|^2 \ge 0$

$$|h'(z)| = \left|1 - \sum_{n=2}^{\infty} n|a_n| z^{n-1}\right| \qquad \ge 1 - \sum_{n=2}^{\infty} n|a_n| r^{n-1} > 1 - \sum_{n=2}^{\infty} n|a_n|$$

$$\ge \sum_{n=1}^{\infty} n|b_n| > \sum_{n=1}^{\infty} n|b_n| r^{n-1} \ge |g'(z)|.$$

if $f(z) \ne 0$, then we show that $f(z_1) \ne f(z_2)$ whenever $z_1 \ne z_2$ .Since Δ is simply connected and convex we have, $z(t) = (1-t)z_1 + t z_2 \in \Delta$, if $0 \le t \le 1, z_1, z_2 \in \Delta$ so that $z_1 \ne z_2$.

$$f(z_1) - f(z_2) = \int_0^1 [(z_2 - z_1)h'(z(t)) + \overline{(z_2 - z_1)g'(z(t))}]dt.$$

Dividing by $z_2 - z_1 \ne 0$, and taking real part

$$\operatorname{Re}\frac{f(z_2)-f(z_1)}{z_2-z_1}=\int_0^1 \operatorname{Re}[h'(z(t))+\frac{\overline{(z_2-z_1)}}{z_2-z_1}g'(z(t))]dt \qquad (5)$$

$$\geq \int_0^1 \operatorname{Re}[h'(z(t))-|g'(z(t))|]dt$$

which implies that

$$\operatorname{Re} h'(z(t))-|g'(z(t))| \geq \operatorname{Re} h'(z)-\sum_{n=1}^{\infty} n|b_n|$$

by (4)

$$\geq 1-\sum_{n=2}^{\infty}\frac{2n^2-1-\beta}{1-\beta}|a_n|-\sum_{n=1}^{\infty}\frac{2n^2+1+\beta}{1-\beta}|b_n| \geq 0$$

using this in (5) shows that f is univalent in Δ.

To show that $f \in L_{\overline{H}}(\beta)$, we need to show that if (4) holds, then

$$\operatorname{Re}\left\{\frac{(1+e^{i\alpha})(z^2h''(z)-\overline{z^2g''(z)}+zh'(z)-\overline{zg'(z)})-e^{i\alpha}(h(z)+\overline{g(z)})}{h(z)+\overline{g(z)}}\right\}=\operatorname{Re}\frac{A(z)}{B(z)} \geq \beta$$

where $z=re^{i\theta}, 0 \leq \theta \leq 2\phi, 0 \leq r<1, 0 \leq \beta<1$ and $A(z)=(1+e^{i\alpha})(z^2h''(z)-\overline{z^2g''(z)}+zh'(z)-\overline{zg'(z)})-e^{i\alpha}(h(z)+\overline{g(z)})$ and $B(z)=h(z)+\overline{g(z)}$

Letting $w=\dfrac{A(z)}{B(z)}$, now it is enough to show that $|1-\beta+w| \geq |1+\beta-w|$, that is

$$|A(z)+(1-\beta)B(z)|-|A(z)-(1+\beta)B(z)| \geq 0$$

Substituting A (z) and B (z) we obtain,

$$=\left|(1+e^{i\alpha})(z^2h''(z)-\overline{z^2g''(z)}+zh'(z)-\overline{zg'(z)})-e^{i\alpha}(h(z)+\overline{g(z)})+(1-\beta)(h(z)+\overline{g(z)})\right|$$
$$-\left|(1+e^{i\alpha})(z^2h''(z)-\overline{z^2g''(z)}+zh'(z)-\overline{zg'(z)})-e^{i\alpha}(h(z)+\overline{g(z)})-(1+\beta)(h(z)+\overline{g(z)})\right|$$

$$\geq (2-\beta)|z|-\sum_{n=2}^{\infty}(2n^2-1-\beta)|a_n||z^n|-\sum_{n=1}^{\infty}(2n^2+1+\beta)|b_n||z^n|$$

$$\geq (2-\beta)|z|-\left\{1-\sum_{n=2}^{\infty}\frac{2n^2-1-\beta}{1-\beta}|a_n|-\sum_{n=1}^{\infty}\frac{2n^2+1+\beta}{1-\beta}|b_n|\right\} \geq 0 \quad by\ (4).$$

Theorem 2 Let $f=h+\overline{g}$ be such that h and g are given by (3). Then $f \in L_{\overline{H}}(\beta)$ if and only if

$$\sum_{n=1}^{\infty}\left[\frac{2n^2-1-\beta}{1-\beta}|a_n|+\frac{2n^2+1+\beta}{1-\beta}|b_n|\right] \leq 2 \qquad (6)$$

where $a_1=1$ and $0 \leq \beta<1$.

Proof The 'if 'part follows from Theorem 1, for the only if part, we show that if $f \notin L_{\overline{H}}(\beta)$ and condition (6) does not hold. The necessary and sufficient condition for $f=h+\overline{g}$ given by (3) to be in $L_{\overline{H}}(\beta)$ is that

$$\operatorname{Re}\left\{(1+e^{i\alpha})\left(\frac{z^2f''(z)+zf'(z)}{f(z)}\right)-e^{i\alpha}\right\} \geq \beta$$

$$\Rightarrow \operatorname{Re}\left\{\frac{(1+e^{i\alpha})(z^2h''(z)-\overline{z^2g''(z)}+zh'(z)-\overline{zg'(z)})-e^{i\alpha}(h(z)+\overline{g(z)})}{h(z)+\overline{g(z)}}\right\} \geq \beta$$

which implies

$$\operatorname{Re}\left\{\frac{\left[\begin{array}{l}1-\beta-\sum_{n=2}^{\infty}[n^2-\beta+e^{i\alpha}(n^2-1)]|a_n|z^{n-1}-\sum_{n=1}^{\infty}n(n-1)|b_n|z\overline{z}^{n-2}-\sum_{n=1}^{\infty}n|b_n|\overline{z}^{n-1}\\ -\beta\frac{\overline{z}}{z}\sum_{n=1}^{\infty}|b_n|\overline{z}^{n-1}-e^{i\alpha}\left(\sum_{n=1}^{\infty}n(n-1)|b_n|z\overline{z}^{n-2}+\sum_{n=1}^{\infty}n|b_n|\overline{z}^{n-1}+\frac{\overline{z}}{z}\sum_{n=1}^{\infty}|b_n|\overline{z}^{n-1}\right)\end{array}\right]}{1-\sum_{n=2}^{\infty}|a_n|z^{n-1}+\frac{\overline{z}}{z}\sum_{n=1}^{\infty}|b_n|\overline{z}^{n-1}}\right\} \geq 0$$

The above condition must hold for all values of z, $|z|=r<1$. Choosing the value z on positive real axis, where $0 \leq z=r<1.$, and since $\operatorname{Re}(-e^{i\alpha}) \geq -|e^{i\alpha}|=-1$, the inequality reduces to

$$\frac{1-\beta-\sum_{n=2}^{\infty}[2n^2-\beta-1]|a_n|r^{n-1}-\sum_{n=1}^{\infty}[2n^2+\beta+1]|b_n|r^{n-1}}{1-\sum_{n=2}^{\infty}|a_n|r^{n-1}+\sum_{n=1}^{\infty}|b_n|r^{n-1}} \geq 0 \qquad (7)$$

If the condition in the equation (6) does not hold then the numerator in (7) is negative for r sufficiently close to 1. This contradicts the condition for $f \in L_{\overline{H}}(\beta)$

III. GROWTH AND DISTORTION THEOREMS

The growth and distortion bounds for the functions in this class is discussed in the following theorems

Theorem 3 If $f \in L_{\overline{H}}(\beta)$, then

$$|f(z)| \leq (1+|b_1|)r+\left[\frac{1-\beta}{7-\beta}-\frac{3+\beta}{7-\beta}|b_1|\right]r^2, |z|=r<1 \qquad (8)$$

and

$$|f(z)| \geq (1-|b_1|)r-\left[\frac{1-\beta}{7-\beta}-\frac{3+\beta}{7-\beta}|b_1|\right]r^2, |z|=r<1 \qquad (9)$$

Proof Let $f \in L_{\overline{H}}(\beta)$, taking absolute value for f

$$|f(z)| \leq (1+|b_1|)r+\sum_{n=2}^{\infty}[|a_n|+|b_n|]r^n$$

$$\leq (1+|b_1|)r+\frac{1-\beta}{7-\beta}\left[1-\frac{3+\beta}{1-\beta}|b_1|\right]r^2$$

$$=(1+|b_1|)r+\left[\frac{1-\beta}{7-\beta}-\frac{3+\beta}{7-\beta}|b_1|\right]r^2.$$

and

$$|f(z)| \geq (1-|b_1|)r-\sum_{n=2}^{\infty}[|a_n|+|b_n|]r^n$$

$$\geq (1-|b_1|)r - \frac{1-\beta}{7-\beta}\left[1-\frac{3+\beta}{1-\beta}|b_1|\right]r^2$$

$$= (1-|b_1|)r - \left[\frac{1-\beta}{7-\beta}-\frac{3+\beta}{7-\beta}|b_1|\right]r^2.$$

Theorem 4 If $f \in L_{\overline{H}}(\beta)$, then

$$|f'(z)| \leq (1+|b_1|) + \left[\frac{1-\beta}{7-\beta}-\frac{3+\beta}{7-\beta}|b_1|\right]r$$

and

$$|f'(z)| \geq (1-|b_1|) - \left[\frac{1-\beta}{7-\beta}-\frac{3+\beta}{7-\beta}|b_1|\right]r, |z| = r < 1.$$

Proof Let $f \in L_{\overline{H}}(\beta)$, taking absolute value for f'

$$|f'(z)| \leq (1+|b_1|) + \sum_{n=2}^{\infty}\left[n|a_n| + n|b_n|\right]r^{n-1}$$

$$\leq (1+|b_1|) + \frac{1-\beta}{7-\beta}\left[1-\frac{3+\beta}{1-\beta}|b_1|\right]r$$

$$= (1+|b_1|) + \left[\frac{1-\beta}{7-\beta}-\frac{3+\beta}{7-\beta}|b_1|\right]r.$$

and

$$|f'(z)| \geq (1-|b_1|) - \sum_{n=2}^{\infty}\left[n|a_n| + n|b_n|\right]r^{n-1}$$

$$\geq (1-|b_1|) - \frac{1-\beta}{7-\beta}\left[1-\frac{3+\beta}{1-\beta}|b_1|\right]r$$

$$= (1-|b_1|) - \left[\frac{1-\beta}{7-\beta}-\frac{3+\beta}{7-\beta}|b_1|\right]r.$$

Theorem 5 $f \in L_{\overline{H}}(\beta)$ if and only if

$$f(z) = \sum_{n=1}^{\infty}(X_n h_n + Y_n g_n) \qquad (10)$$

where

$$h_1(z) = z, h_n(z) = z - \frac{1-\beta}{2n^2-1-\beta}z^n, (n=2,3...),$$

$$g_n(z) = z + \frac{1-\beta}{2n^2+1+\beta}\overline{z}^n, (n=1,2,3...)$$

$\sum_{n=1}^{\infty}(X_n + Y_n) = 1, X_n \geq 0$ and $Y_n \geq 0$

Proof

$$f(z) = \sum_{n=1}^{\infty}(X_n h_n + Y_n g_n) = \sum_{n=1}^{\infty}X_n h_n + \sum_{n=1}^{\infty}Y_n g_n$$

$$= z - \sum_{n=2}^{\infty}\frac{1-\beta}{2n^2-1-\beta}X_n z^n + \sum_{n=1}^{\infty}\frac{1-\beta}{2n^2+1+\beta}Y_n\overline{z}^n$$

then

$$\sum_{n=2}^{\infty}\left[\frac{2n^2-1-\beta}{1-\beta}\right]\left[\frac{1-\beta}{2n^2-1-\beta}\right]X_n + \sum_{n=1}^{\infty}\left[\frac{2n^2+1+\beta}{1-\beta}\right]\left[\frac{1-\beta}{2n^2+1+\beta}\right]Y_n$$

$= 1 - X_1 \leq 1$, and hence $f \in L_{\overline{H}}(\beta)$

Conversely,

$f \in L_{\overline{H}}(\beta)$,set $X_n = \frac{2n^2-1-\beta}{1-\beta}|a_n|, (n=1,2...)$ and $Y_n = \frac{2n^2+1+\beta}{1-\beta}|b_n|, (n=1,2...)$ then $\sum_{n=1}^{\infty}(X_n + Y_n) \leq 1$ by Theorem 2, $0 \leq X_n \leq 1, (n=1,2...)$ and $0 \leq Y_n \leq 1, (n=1,2...)$ consequently we have $f(z) = \sum_{n=1}^{\infty}(X_n h_n + Y_n g_n)$.

IV. CONVOLUTION

Definition For $f(z) = z - \sum_{n=2}^{\infty}a_n z^n + \sum_{n=1}^{\infty}b_n\overline{z}^n$ and $F(z) = z - \sum_{n=2}^{\infty}A_n z^n + \sum_{n=1}^{\infty}B_n\overline{z}^n$ the modified Hadamard product of two harmonic functions f and F is defined as

$$(f*F) = f(z)*F(z) = z - \sum_{n=2}^{\infty}a_n A_n z^n + \sum_{n=1}^{\infty}b_n B_n\overline{z}^n \qquad (11)$$

Theroem 6 For $0 \leq \gamma \leq \beta < 1$, let $f \in L_{\overline{H}}(\gamma)$ and $F \in L_{\overline{H}}(\beta)$, then $f*F \in L_{\overline{H}}(\gamma) \subset L_{\overline{H}}(\beta)$.

Proof Let $f(z) = z - \sum_{n=2}^{\infty}a_n z^n + \sum_{n=1}^{\infty}b_n\overline{z}^n$ be in $L_{\overline{H}}(\gamma)$ and $F(z) = z - \sum_{n=2}^{\infty}A_n z^n + \sum_{n=1}^{\infty}B_n\overline{z}^n$ be in $L_{\overline{H}}(\beta)$, the coefficients of $f*F$ is given by

$$\sum_{n=1}^{\infty}\left[\frac{2n^2-1-\beta}{1-\beta}|a_n A_n| + \frac{2n^2+1+\beta}{1-\beta}|b_n B_n|\right]$$

$$\leq \sum_{n=1}^{\infty}\left[\frac{2n^2-1-\beta}{1-\beta}|a_n| + \frac{2n^2+1+\beta}{1-\beta}|b_n|\right].$$

because $f \in L_{\overline{H}}(\gamma)$. Hence we have $f*F \in L_{\overline{H}}(\beta)$.

Theorem 7 The family $L_{\overline{H}}(\beta)$ is closed under convex combination.

Proof For i=1,2… let $f_i \in L_{\overline{H}}(\beta)$ where f_i is given by

$$f_i(z) = z - \sum_{n=2}^{\infty}|a_{in}|z^n + \sum_{n=1}^{\infty}|b_{in}|\overline{z}^n.$$

for $0 \leq t_i \leq 1, \sum_{i=1}^{\infty}t_i = 1$, the convex combination of f_i is

$$\sum_{i=1}^{\infty}t_i f_i(z) = z - \sum_{n=2}^{\infty}\left(\sum_{i=2}^{\infty}t_i|a_{in}|z^n\right) + \sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty}t_i|b_{in}|\overline{z}^n\right)$$

$$= \sum_{i=1}^{\infty}t_i\left\{\sum_{n=1}^{\infty}\left[\frac{2n^2-1-\beta}{1-\beta}|a_{in}| + \frac{2n^2+1+\beta}{1-\beta}|b_{in}|\right]\right\}$$

$$\leq 2\sum_{i=1}^{\infty}t_i = 2$$

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