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Smale Horseshoes and Ši'lnikov Chaos in a New Three-Order Dynamical System

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Abstract- This paper presents the existence of Ši'lnikov homoclinic and heteroclinic orbits in the novel system by using the undetermined coefficient method. The Ši'lnikov criterion along with some technical conditions guarantees that the novel system has both Smale horseshoes and the horseshoe chaos. Moreover, it is shown that the heteroclinic and homoclinic orbits together determine the geometric structure of attractors.

Keywords- *Ši'lnikov Criterion, Novel System, Homoclinic and Heteroclinic Orbits, Smale Horseshoes, Undetermined Coefficients Method*

I. INTRODUCTION

As one of the most fascinating nonlinear phenomena, chaos has been extensively studied in the field of mathematics, physics, astronomy, chemistry and engineering communities in the last four decades. An extremely remarkable finding is that chaos has great potential applications in many technological and engineering disciplines [1-14]. As a result, many efforts have been devoted to the generation of complex chaotic dynamics in the continuous-time systems [15-20].

Zhou and Chen [18] proposed a new analysis tool, i.e., undetermined coefficient method, which is a powerful tool to determine the existence of Ši'lnikov chaos, and has been successfully used in the literature [17, 18, 21–24] to construct the heteroclinic or homoclinic orbits of Ši'lnikov type. This work find heteroclinic and homoclinic orbits of Ši'lnikov type at the two equilibrium points of the novel system.

This paper is organized as following: Section 2, some basic concepts and terminologies related to homoclinic and heteroclinic orbits are reviewed. In Section 3 the Ši'lnikov heteroclinic orbits of the novel system is studied in detail by using the undetermined coefficient method. In this section, the algebraic expression of the heteroclinic orbit will also be derived, and the uniform convergence of its series expansion is proved. Section 4 introduces the undetermined coefficient method, which will be used to find homoclinic orbits in the novel system. Also in this section, the algebraic expression of homoclinic orbits will also be derived, and the uniform convergence of its series expansion is proved. Finally, some concluding remarks will be provided in section 5.

II. HOMOCLINIC AND HETEROCLINIC ORBIT

Consider the third-order autonomous system

$$\frac{dx}{dt} = f(x), \quad x \in R^3 \tag{1}$$

where the vector field $f(x): R^3 \longrightarrow R^3$ belongs to class $C^r (r \geq 2)$.

Let $x_e \in R^3$ be an equilibrium point of system (1). Then x_e is called a hyperbolic saddle focus (or simply, saddle focus) if the eigenvalues of the Jacobian $A = Df$, evaluated at x_e, are:

$$\gamma,\ \rho \pm i\,\omega, \rho\gamma < 0, \omega \neq 0 \tag{2}$$

where γ, ρ and ω are real.

A homoclinic orbit $\gamma(t)$ refers to a bounded trajectory of system (1) that is doubly asymptotic to an equilibrium point P of the system, i.e. $\lim_{t\to+\infty} \gamma(t) = \lim_{t\to-\infty} \gamma(t) = P$.

A heteroclinic orbit $\delta(t)$, is similarly defined except that there are two distinct saddle-focus P_1 and P_2, being connected by the orbit, one corresponding to the forward asymptotic time, and the other, to the reverse asymptotic time limit, $\lim_{t\to+\infty} \delta(t) = P_1$ and $\lim_{t\to-\infty} \delta(t) = P_2$.

The heteroclinic or homoclinic Ši'lnikov method, namely, the Ši'lnikov criterion for the existence of chaos, is summarized in the following theorems [25-29].

A. Theorem 1[the heteroclinic Ši'lnikov theorem]

Suppose that two distinct equilibrium points, denoted by χ_e^1 and χ_e^2, respectively, of system (1) are saddle foci, whose characteristic values γ_k and $\rho_k \pm i\,\omega_k$ (k =1,2) satisfy the following Ši'lnikov inequalities:

$$|\gamma_k| > |\rho_k| > 0, \quad k = 1,2 \quad , \omega \neq 0 \tag{2}$$

Under constraint

$$\rho_1 \rho_2 > 0 \qquad or \qquad \gamma_1\gamma_2 > 0 \tag{3}$$

Suppose also that there exists a heteroclinic orbit joining χ_e^1 and χ_e^2, then:

(i) The Ši'lnikov map, defined in a neighborhood of the heteroclinic orbit, has a countable number of Smale horseshoes in its discrete dynamics;

(ii) For any sufficiently small C^1-perturbation g of f, the perturbed system

$$\frac{dx}{dt} = g(x), \quad x \in R^3 \tag{4}$$

has at least a finite number of Smale horseshoes in the discrete dynamics of the Ši'lnikov map defined near the heteroclinic orbit;

(iii) Both the original system (1) and the perturbed system (4) have horseshoe type of chaos.

B. Theorem 2 [the homoclinic Ši'lnikov theorem]

Suppose that one equilibrium point of system (1), denoted by χ_e ,is saddle focus, whose eigenvalues γ and $\rho \pm i\omega$ satisfy the following Ši'lnikov condition:

$$\gamma\rho < 0, \quad |\gamma| > |\rho| > 0, \quad , \omega \neq 0 \tag{5}$$

Suppose also that there exists a homoclinic orbit connecting χ_e then:

(i) The Ši'lnikov map, defined in a neighborhood of the homoclinic orbit of the system, possesses a countable number of Smale horseshoes in its discrete dynamics;

(ii) For any sufficiently small C^1-perturbation g of f, the perturbed system

$$\frac{dx}{dt} = g(x), \quad x \in R^3 \tag{6}$$

has at least a finite number of Smale horseshoes in the discrete dynamics of the Ši'lnikov map defined near the homoclinic orbit;

(iii) Both the original system (1) and the perturbed system (6) exhibit horseshoe type of chaos.

For convenience, a homoclinic or heteroclinic orbit satisfying (2) or (3) and (5) is referred to as the Si'lnikov type. Thus, the homoclinic or heteroclinic Si'lnikov criterion implies that if system (1) has one homoclinic or heteroclinic orbit of the Si'lnikov type, which connects a saddle focus of the system to itself or two distinct saddle foci of the system, then it has Smale horseshoe chaos, which is rigorous in mathematical sense.

III. THE HETEROCLINIC ORBITS OF THE NOVEL SYSTEM

The novel system [30] can be described by the following differential equation:

$$\frac{dx}{dt} = y - x$$
$$\frac{dy}{dt} = a.y - xz \tag{7}$$
$$\frac{dz}{dt} = xy - b$$

where $a,b \in R^+$. The novel system (7) has chaotic attractors as shown in Fig.1 when $a = 0.5, b = 0.5$. The system has two equilibrium points: $E_{1,2} = (\pm\sqrt{b}, \pm\sqrt{b}, a)$

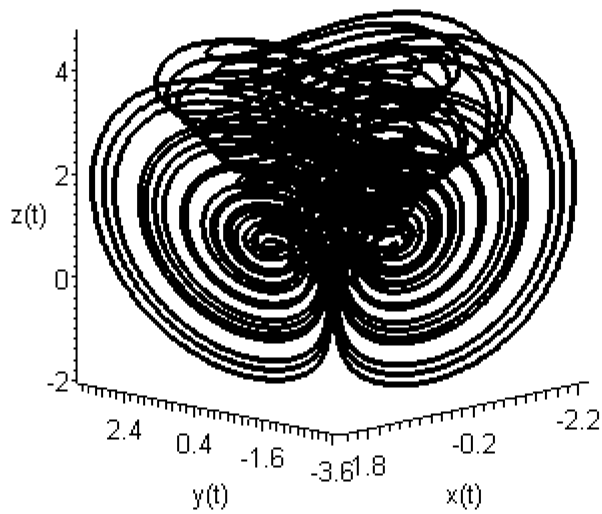

Figure 1. Phase portraits of the novel system in the three-dimensional

Then, the characteristic equation of the system (7) at the points E_1 and E_2 is:

$$\lambda^3 + (1-a)\lambda^2 + b\lambda + 2b = 0 \tag{8}$$

If $a < 1$ Due to Descartes' rule of signs [31, 32]. The characteristic equation (8) has no positive real root. Thus, it has at least one negative real root.

In equation (8) Let $\lambda = \mu - \frac{(1-a)}{3}$, then equation (10) becomes:

$$\mu^3 + p\mu + q = 0 \tag{9}$$

where

$$p = b - \frac{(1-a)^2}{3}, q = \frac{2(1-a)^3}{27} - \frac{b(1-a)}{3} + 2b \tag{10}$$

and $\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3$

By Cardan formula, the equation (9) has a unique negative real root, α , and a conjugate pair of complex roots, $\rho \pm i\,\omega$, where

$$\alpha = \sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta}},$$

$$\rho = -\frac{1}{2}\left(\sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta}}\right)$$

$$\omega = \frac{\sqrt{3}}{2}\left(\sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}} - \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta}}\right)$$

When $\Delta > 0$, therefore, the algebraic equation (9) has the following three roots:

$$\lambda_1 = -\frac{(1-a)}{3} + \alpha, \qquad \lambda_{2,3} = -\frac{(1-a)}{3} + \rho \pm \omega i \tag{11}$$

Respectively, where $\lambda_1 < 0$.To ensures that the real part of the complex conjugate roots is positive and it is further required that:

$$-\sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}} - \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta}} > \frac{2(1-a)}{3} > 0 \tag{12}$$

Then, when $\Delta > 0$ and inequality (12), one can easily obtain that the two points E_1 and E_2 are of hyperbolic saddle foci type.

A. The existence of heteroclinic orbits in the novel system

In this part, we will investigate the undetermined coefficient method to prove the existence of heteroclinic orbits of system (7).

From (7), we find that:

$$\begin{cases} y = \dot{x} + x & , \dot{y} = \ddot{x} + \dot{x}, \quad \ddot{y} = \dddot{x} + \ddot{x} \\ z = \dfrac{(ay - \dot{y})}{x}, & \dot{z} = \dfrac{(a\dot{y} - \ddot{y})x - \dot{x}(ay - \dot{y})}{x^2} \end{cases} \tag{13}$$

Substituting (13) into the third equation of system (7) gives

$$x\left(\dddot{x} + (1-a)\ddot{x}\right) - \dot{x}(\ddot{x} + (1-a)\dot{x} + x^3) + bx^2 - x^4 = 0 \tag{14}$$

If $x(t)$ is found, then $z(t)$ and $y(t)$ will also be determined. Therefore, finding the heteroclinic orbit of system (7) is now reduced to seeking a function $\psi(t)$ such that $\psi(t) = x(t)$ satisfying (14) and

$$\psi(t) \longrightarrow -\sqrt{b}, \qquad as \quad t \longrightarrow +\infty$$
$$\psi(t) \longrightarrow \sqrt{b}, \qquad as \quad t \longrightarrow -\infty$$

or

$$\psi(t) \longrightarrow \sqrt{b}, \qquad as \quad t \longrightarrow +\infty$$
$$\psi(t) \longrightarrow -\sqrt{b}, \qquad as \quad t \longrightarrow -\infty$$

Without loss of generality, one may stipulate a definite direction as follows: from E_1 to E_2 corresponds to $t \to +\infty$, while from E_2 to E_1 corresponds to $t \to -\infty$. Let

$$x(t) = \psi(t) = -\delta + \sum_{k=1}^{\infty} d_k\, e^{k\alpha t} \le \tag{15}$$
$$\delta = \sqrt{b}\text{ , and } t > 0$$

Where $\alpha < 0$, is an undetermined constant and d_k $(k \ge 1)$ are undetermined coefficient.

Substituting (15) into Eq. (14), we get:

$$\sum_{k=1}^{\infty} \delta\left(G(\alpha k)\right) d_k\, e^{\alpha kt} = H_1 + H_2 + H_3 \tag{16}$$

Where:

$$H_1 = \sum_{k=2}^{\infty}\left(\sum_{i=1}^{k-1}\left(\alpha^3 i^3 + (1-a)\alpha^2 i^2 + (3b)\alpha i\right) + \left(\left(\alpha^2 i^2 + (1-a)\alpha i\right)(k-i)\alpha + \delta^2(3+6a)\right)\right) d_i d_{k-i} e^{\alpha kt},$$

$$H_2 = -\delta \sum_{k=3}^{\infty}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1}(4\delta + 3\delta\alpha(k-j))d_i d_{j-i} d_{k-j} e^{\alpha kt},$$

$$H_3 = \sum_{k=4}^{\infty}\sum_{m=3}^{k-1}\sum_{j=2}^{m-1}\sum_{i=1}^{j-1}(1 + (k-m)\alpha) d_i d_{j-i} d_{m-j} d_{k-m} e^{\alpha kt}.$$

and

$$G(\alpha k) = (\alpha^3 k^3 + (1-a)\alpha^2 k^2 + b\alpha k + 2b) \tag{17}$$

Comparing the coefficients of $e^{\alpha kt}$ $(k \ge 1)$ of the same power terms, we obtain the following results.

For $k = 1$,

$$\alpha^3 + (1-a)\alpha^2 + b\alpha + 2b = 0 \tag{18}$$

Which is just the characteristic polynomial of the Jacobian of the linearized equation of system (7) evaluated at the equilibrium point E_1 or E_2. Since (8) has the unique negative root for given parameters, there exist a $\alpha < 0$ such that $G(\alpha) = 0$, and for $k > 1$,

$$G(\alpha k) = (\alpha^3 k^3 + (1-a)\alpha^2 k^2 + b\alpha k + 2b) \neq 0 \quad , k > 1$$

So, for $k = 2$,

$$d_2 = a_1^2 (H_4) / \delta G(2\alpha) \tag{19}$$

where

$$H_4 = \left(\alpha^3 + (1-a)\alpha^2 + (3b)\alpha\right) + \left(\left(\alpha^2 + (1-a)\alpha\right)\alpha + \delta^2(3+6a)\right)$$

For $k = 3$,

$$d_3 = \frac{[H_5 + H_6]}{\delta G(3\alpha)} \tag{20}$$

Where

$$H_5 = \sum_{i=1}^{2} \frac{\left(\alpha^3 i^3 + (1-a)\alpha^2 i^2 + (3b)\alpha i\right) +}{\left(\left(\alpha^2 i^2 + (1-a)\alpha i\right)(k-i)\alpha + \delta^2(3+6a)\right)} d_i d_{3-i}$$

$$H_6 = -\delta(4\delta + 3\delta\alpha) d_1^3$$

Finally, for $k \geq 4$,

$$d_k = \frac{[H_7 + H_8 + H_9]}{\delta G(\alpha k)} \tag{21}$$

where:

$$H_1 = \sum_{i=1}^{k-1} \left(\alpha^3 i^3 + (1-a)\alpha^2 i^2 + (3b)\alpha i\right) + \left(\left(\alpha^2 i^2 + (1-a)\alpha i\right)(k-i)\alpha + \delta^2(3+6a)\right) d_i d_{k-i},$$

$$H_2 = -\delta \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} (4\delta + 3\delta\alpha(k-j) d_i d_{j-i} d_{k-j},$$

$$H_3 = \sum_{m=3}^{k-1} \sum_{j=2}^{m-1} \sum_{i=1}^{j-1} (1 + (k-m)\alpha) d_i d_{j-i} d_{m-j} d_{k-m}.$$

So α is completely determined by *a*, *b* and *c*, and d_k $(k \geq 2)$ is completely determined by a, b, c, α.

To this point, the first part of the heteroclinic orbit corresponding to $t > 0$ has been determined, see Fig.2. Next, its second part corresponding to $t < 0$ will be constructed.

From the continuity of the solution, we have:

$$\sum_{k=1}^{\infty} d_k = \delta \tag{22}$$

Which will determine the value of d_1.

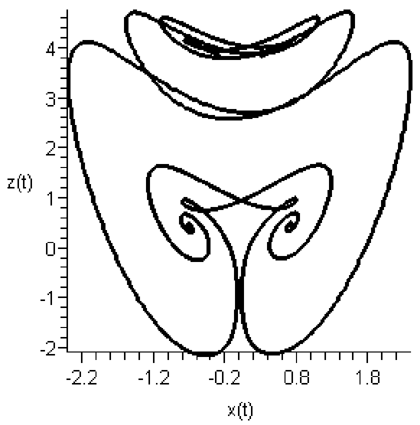

Figure 2. Numerical approximation of the heteroclinic and homoclinic orbit joining E_1 and E_2 in the novel system with parameter values: a =0.5,b =0.5 .

B. The uniform convergence of heteroclinic orbits series expansion

The uniform convergence of the series expansion (15) of the heteroclinic orbit is investigated. For simplicity, we only consider the case in which system (7) has the special parameter set that generates two-scroll attractors. For other parameter sets, the proof is similar if the heteroclinic orbit exists. Since the value of α and d_1 can be determined from (18)-(22). So, d_k $(k \geq 4)$ can also, be determined, with $\sum_{k=1}^{\infty} d_k = \delta$. When $a = 0.5$, $b = 0.5$ and $\delta = \sqrt{b}$, the values of α and d_k can be determined by (18)-(21). and (22) as, $|d_2| = 0.1178511302 d_1^2$, $|d_1^3|$, $|d_4| = 0.008258268791 d_1^4$, one can inductivity prove that $|d_k| < 10^{-k+2} \; |d_1^k|$, $(k \geq 4)$

We need to seek d_1 with $\sum_{k=1}^{\infty} d_k = \delta$. Numerical simulation shows that a "stable" d_1 indeed exist near 5.020592 with relative error no greater than 1%. So when $(k \geq 4)$ d_k is bounded, that is there exists an $l > 0$, such that $|d_k| \leq l$, $k = 1, 2, \ldots\ldots$, consequently, $\sum_{k=1}^{\infty} \left| d_k e^{\alpha k t} \right| \leq l$

$\sum_{k=1}^{\infty} d_k e^{\alpha kt}$ is convergent on $(0, +\infty)$. So $-\delta + \sum_{k=1}^{\infty} d_k e^{\beta kt}$ is convergent on $(0, +\infty)$. similarly, the convergence of $\delta - \sum_{k=1}^{\infty} d_k e^{-\beta kt}$ on $(-\infty, 0)$ can be also proved.

Finally, due to Ši'lnikov criterion, one may impose the following condition:

C. Theorem 3

If $\Delta > 0$ and (12) are satisfied, then the system (7) has one Ši'lnikov heteroclinic orbit and the corresponding chaos is of horseshoe type**.**

Obviously, the typical parameters a =0.5,b =0.5 are always satisfied. So there exist heteroclinic orbits of Ši'lnikov type, and as a result, there exist a countable number of Smale horseshoes. Therefore, there exists an invariant set constituting the complex attractor. That is the essence of the geometric structure of the attractor.

IV. THE EXISTENCE OF HOMOCLINIC ORBITS

In this part, we will investigate the undetermined coefficient method to prove the existence of homoclinic orbits of system (7).

From (7), we find that:

$$y = \frac{\dot{z} + b}{x}, \dot{y} = \frac{(\ddot{z})x - \dot{x}(\dot{z} + b)}{x^2} \tag{23}$$

Substituting (23) into the first equation of system (7) gives

$$x.\frac{dx}{dt} + x^2 = \dot{z} + b \tag{24}$$

Then, solving (24) yields:

$$x^2(t) = e^{-2t}\left[\int_0^t (2b + 2\dot{z}(s))e^{2s}\, ds + c_0\right] \tag{25}$$

Next, substituting (23) and (25) into the second equation of system (7) leads to

$$zH_1^2 = H_1((a-1)\dot{z} + ab - b - \ddot{z}) + (\dot{z} + b)^2 \tag{26}$$

Where

$$H_1 = e^{-2t}\left[\int_0^t (2b + 2\dot{z}(s))e^{2s}\, ds + c_0\right] \tag{27}$$

If $z(t)$ is found, then $x(t)$ and $y(t)$ will also be determined. Therefore, finding the homoclinic orbit of system (7) is now reduced to seeking a function $\zeta(t)$ such that $\zeta(t) = z(t)$ satisfying (26) and

$$\zeta(t) \longrightarrow a, \qquad as \quad t \longrightarrow \pm\infty$$

Let:

$$z(t) = \zeta(t) = a + \sum_{k=1}^{\infty} d_k\, e^{kdt}, t > 0 \tag{28}$$

and

$$c_0 = b + 2\sum_{k=1}^{\infty} \frac{\alpha k}{\alpha k + 2} d_k$$

Where $\alpha < 0$, is an undetermined constant and $(d_k \geq 1)$ are undetermined coefficient, Then

$$H_1 = b + 2\left(\sum_{k=1}^{\infty} \frac{\alpha k}{\alpha k + 2} d_k\right) e^{\alpha kt} \tag{29}$$

Substituting in the equation (26)

Then, (28) can be written in the following form:

$$\sum_{k=1}^{\infty} G(\alpha k) d_k e^{\alpha kt} = H_1 + H_2 \tag{30}$$

where

$$H_1 = \left(2\sum_{k=1}^{\infty} \frac{\alpha k}{\alpha k + 2} d_k\, e^{k\alpha t}\right)\left(\sum_{k=1}^{\infty}\left((a-1)\alpha k - \alpha^2 k^2\right) a_k e^{\alpha kt}\right) +$$
$$\sum_{k=2}^{\infty} \sum_{i=1}^{k-1} \alpha i\, d_i\, d_{k-i}\, e^{k\alpha t} - 4\left(\sum_{k=2}^{\infty}\sum_{i=1}^{k-1}\left(\frac{\alpha i\,(b+a)}{(\alpha k + 2)}\right) d_i\, d_{k-i}\, e^{\alpha kt}\right)$$
$$H_2 = -4\sum_{k=1}^{\infty} d_k\, e^{k\alpha t}\left(\sum_{k=2}^{\infty}\sum_{i=1}^{k-1}\left(\frac{(\alpha i)}{(\alpha k + 2)}\right) d_i\, d_{k-i}\, e^{\alpha kt}\right)$$

and

$$G(\alpha k) = \frac{(ab + b)2\alpha k}{\alpha k + 2} - b((a-1)\alpha k + 2\alpha k - -b - \alpha^2 k^2)$$

Comparing the coefficients of $e^{k\alpha t}$ $(k \geq 1)$ of the same power terms, we obtain the following results. For $k = 1$,

$$G(\alpha) = \alpha^3 + (1-a)\alpha^2 + b\alpha + 2b = 0 \tag{31}$$

Which is just the characteristic polynomial of the Jacobian of the linearized equation of system (7) evaluated at both of equilibrium points $E_{1,2}$.

Since (8) has the unique negative root for given parameters, there exist a $\alpha < 0$ such that $F(\alpha) = 0$, and for $k > 1$,

$$F(\alpha k) = (\alpha k)^3 + (1-a)(\alpha k)^2 + b\alpha k + 2b \neq 0$$

That is

$$G(\alpha k) = (\alpha k)^3 + (1-a)(\alpha k)^2 + b\alpha k + 2b \quad , k > 1 \otimes$$

So, for $k = 2$,

$$d_2 = [H_3] / G(2\alpha)$$

$$H_3 = \left(2\sum_{k=1}^{2}\frac{\alpha k}{\alpha k + 2} d_k\right)\left(\sum_{k=1}^{2}\left((a-1)\alpha k - \alpha^2 k^2\right) d_k\right)$$

$$+\sum_{i=1}^{2}\alpha i\, d_i\, d_{k-i} - 4\sum_{i=1}^{2}\left(\frac{\alpha i\,(b+a)}{(\alpha k + 2)}\right) d_i\, d_{k-i},$$

for $k \geq 3$,

$$d_k = \left[H_4 + H_5\right] / G(k\alpha)$$

where

$$H_4 = \left(2\sum_{k=1}^{k}\frac{\alpha k}{\alpha k + 2} d_k\right)\left(\sum_{k=1}^{k}\left((a-1)\alpha k - \alpha^2 k^2\right) a_k\right)$$

$$+\sum_{i=1}^{k-1}\alpha i\, d_i\, d_{k-i} - 4\left(\sum_{i=1}^{k-1}\left(\frac{\alpha i\,(b+a)}{(\alpha k + 2)}\right) d_i\, d_{k-i}\right)$$

$$H_5 = -4\left(\sum_{j=2}^{k-1}\sum_{i=1}^{j-1}\left(\frac{(\alpha i)}{(\alpha k + 2)}\right) d_i\, d_{k-i}\right)$$

where, so α is completely determined by a, b and c, and d_k $(k \geq 2)$ is completely determined by $a, b,$ α and d_1.

Due to the symmetry of the system, one component of the homoclinic orbit of (7) has the following form:

$$\zeta(t) = \begin{cases} a + \sum_{k=1}^{\infty} d_k e^{\alpha k t}, & for \quad t > 0 \\ or & \\ a - \sum_{k=1}^{\infty} d_k e^{-\alpha k t}, & for \quad t < 0 \end{cases} \qquad (33)$$

From the continuity of the solution, we have:

$$\sum_{k=1}^{\infty} d_k = a \qquad (34)$$

Which will determine the value of d_1.

A. The uniform convergence of homoclinic orbits series expansion

The uniform convergence of the series expansion (33) of the homoclinic orbit is investigated. For simplicity, we only consider the case in which system (7) has the special parameter set that generates two-scroll attractors. For other parameter sets, the proof is similar if the heteroclinic orbit exists.

B. Theorem 4

If $\Delta > 0$ and (14) are satisfied, then the system (7) has one Ši'lnikov homoclinic orbit of which one component has the form (33), and the corresponding chaos is of horseshoe type.

Note that: $\Delta > 0$ and (14) are satisfied are obviously satisfied when $a = 0.5$, $b = 0.5$ with the theorem 1 and theorem 2.

V. Conclusions

Using the undetermined coefficient method, the existence of two types of orbits in the novel system, i.e., heteroclinic and homoclinic orbits with the explicit and uniformly convergent algebraic expressions, has been proved ((the heteroclinic and homoclinic orbits are exist at E_1 and E_2)). By the Ši'lnikov criterion, the novel system has the Smale horseshoe chaos.

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