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A Special Characterization for Joachimsthal and Terquem Type Theorems

Filiz Ertem Kaya
Department of Mathematics, Faculty of Science and Arts, University of Omer Halisdemir, Campus, Nigde, Turkey
(fertem@ohu.edu.tr)

Abstract- In this paper, we investigate the eneryg of two curves on different surfaces and strips in type of curvatures of strips at first time. We also observe some characterizations of finding energy of the curves on spherical helix strip by using Terquem Theorem (one of the Joachimsthal Theorems).

Keywords- *Curve-surface pair (strip), Curvature, Energy, Joachimsthal Theorem*

I. INTRODUCTION

The word ‘energy’ comes from energeia in Greek. First occurred in the studies of Aristoteles in 4th century B.C..

The ‘energy’ term came from by defining Gottfried Leibniz that was vis viva (live force). Leibniz defined vis viva that is the multiplying matter's mass and its squared velocity.

In 1807, Thomas Young used energy term as meaning of today instead of vis visa that was the first person. Gustave-Gaspard Coriolis defined kinetic energy in 1829; William Rankine defined potential energy in 1853 as today's meanings.

Energy is not only used by physicists but also by mathematicians. For example in Differential Geometry Horn found the curve which passes through two specified points with specified orientation while minimizing.

$$\varepsilon = \int \kappa^2 ds$$

In this formulation κ is the curvature and s the arc distance. Horn found interesting applications in 1983. He applied and introduced the energy with Differential Geometry at first. In a thin beam, curvature at a point is proportional to the bending moment [11,12]. The total elastic energy stored in a thin beam is therefore proportional to the integral of the square of the curvature [11,12]. The shape taken on by a thin beam is the one which minimizes its internal strain energy. This is why we call the curve sought here the minimum energy curve. A thin metal or wooden strip used by a draftsman to smoothly connect a number of points is called a spline [11,12]. Such splines are used in creating lofted surfaces from plane.

In addition of the finding energy of the curve, Horn found a semicircle, two circular arcs, the best ellipse, three, four, five, six and more arcs. So Horn’s paper provides to obtain this new knowledge and there are conservation laws in physics.

In this paper, the energy of curves ε_1 and ε_2 is investigated by its curvatures of the strips and find some relations and characterizations between ε_1 and ε_2 on Joachimsthal Theorems at first time like Horn’s method.

II. PRELIMINARIES

We now review some basic concepts on classical differential geometry of space curves in Euclidean space, general helix and slant helix. Let $\alpha : I \to R^3$ be a curve $\alpha'(s) \neq 0$ where $T(s) = \alpha'(s)$ is a unit tangent vector of α at s and M be a surface in Euclidean 3-space. We define a surface element of M is the part of a tangent plane at the neighbour of the point. The locus of the these surface element along the curve is called a curve-surface pair as shown (α, M). We study this Euclidean Space, may study in Minkowski space and rotational surfaces. See more details in [7, 13].

A. *The Curve-Surface Pair (Strip)*

Definition: Let M and α be a surface in E^3 and a curve in $M \subset E^3$. We define a surface element of M is the part of a tangent plane at the neighbour of the point. The locus of these surface element along the curve α is called a curve-surface pair and is shown as (α, M).

Definition: Let $\{\vec{t}, \vec{n}, \vec{b}\}$ and $\{\vec{\xi}, \vec{\eta}, \vec{\zeta}\}$ be the curve and curve-surface pair’s vector fields. The curve-surface pair’s tangent vector field, normal vector field and binormal vector field is given by $\vec{t} = \vec{\xi}, \vec{\zeta} = \vec{N}$ = ($\vec{N} = \vec{n}$) and $\vec{\eta} = \vec{\zeta} \wedge \vec{\xi}$ ([1-6,8-10]).

1) *Curvatures of the Curve-Surface Pair and Curvatures of the Curve*

Let $k_n = -b, k_g = c, t_r = a$ and $\{\vec{\xi}, \vec{\eta}, \vec{\zeta}\}$ be the normal curvature, the geodesic curvature, the geodesic torsion of the strip and the curve-surface pair’s vector fields on α [1-6,9,10].

Then we have:

$$\vec{\xi}' = c\vec{\eta} - b\vec{\zeta}$$
$$\vec{\eta}' = -c\vec{\xi} + a\vec{\zeta} \qquad (1)$$
$$\vec{\zeta}' = b\vec{\xi} - a\vec{\eta}$$

We know that a curve α has two curvatures κ and τ. A curve has a strip and a strip has three curvatures k_n, k_g and t_r.

Let k_n, k_g and t_r be the $-b, c$ and a. From last equations we have $\vec{\xi}' = c\vec{\eta} - b\vec{\zeta}$. If we substitude $\vec{\xi} = \vec{t}$ in last equation, we obtain

$$\vec{\xi}' = \kappa\vec{n}$$

and

$$b = -\kappa \sin\varphi$$
$$c = \kappa \cos\varphi$$

([2-6,9,10]) From last two equations we obtain:

$$\kappa^2 = b^2 + c^2$$

This equation is a relation between the curvature κ of a curve α and normal curvature and geodesic curvature of a curve-surface pair.

By using similar operations, we obtain a new equation as follows

$$\tau = a + \frac{b'c - bc'}{b^2 + c^2}$$

([2-6,9,10]). This equation is a relation between τ (torsion or second curvature of α and curvatures of a curve-surface pair that belongs to the curve α). And also we can write

$$a = \varphi' + \tau.$$

The special case:

If φ is constant, then $\varphi' = 0$. So the equation is $a = \tau$. That is, if the angle is constant, then torsion of the curve-surface pair is equal to torsion of the curve.

Definition: Let α be a curve in $M \subset E^3$. If the geodesic curvature (torsion) of the curve α is equal to zero, then the curve-surface pair (α, M) is called a curvature curve-surface pair (strip) ([2-6,9,10]).

III. GENERAL HELIX

Definition: Let α be a curve in E^3 and V_1 be the first Frenet vector field of α. $U \in \chi(E^3)$ be a constant unit vector field.

If:

$$\langle V_1, U \rangle = \cos\phi \text{ (Constant)}$$

α, φ and $Sp\{U\}$ are called a general helix, the slope and the slope axis ([1,2,6]).

Definition: A regular curve is called a general helix if its first and second curvatures κ and τ are not constant but $\frac{\kappa}{\tau}$ is constant ([1,6]).

Definition: A curve is called a general helix or cylindrical helix it its tangent makes a constant angle with a fixed line in space. A curve is a general helix if and only if the ratio $\frac{\kappa}{\tau}$ is constant ([5,9,12]).

Definition: A helix is a curve in 3-dimensional space. The following parameterization in Cartesian coordinates defines a helix, see [7].

$$x(t) = \cos t$$
$$y(t) = \sin t$$
$$z(t) = t$$

As the parameter t increases $(x(t), y(t), z(t))$ traces a right-handed helix of pitch 2π and Radius 1 about the z axis, in a right-handed coordinate system. In cylindrical coordinates (r, θ, h) the same helix is parameterized by

$$r(t) = 1,$$
$$\theta(t) = t,$$
$$h(t) = t$$

Definition: If the curve α is a general helix, the ratio of the first curvature of the curve to the torsion of the curve must be the constant. The ratio $\frac{\tau}{\kappa}$ is called first harmonic curvature of the curve and is denoted by H_1 or H.

Theorem 3.1: A regular curve $\alpha \subset E^3$ is a general helix if and only if $H(s) = \frac{k_1}{k_2} = const$ for $\forall s \in I$, see [7].

Proof: $(\Rightarrow)$ Let α be a general helix. The slope axis of the curve α is showed $Sp\{U\}$. Note that

$$\langle \alpha'(s), U \rangle = \cos\varphi = const.$$

If the Frenet Threshold is V_1, V_2, V_3 at the point $\alpha(s)$, then we have

$$\langle V_1(s), U \rangle = \cos\varphi.$$

If we take derivative of the both sides of the last equation, then we have

$$\langle k_1 V_2(s), U \rangle = 0 \Rightarrow \langle V_2(s), U \rangle = 0.$$

Hence

$U \in Sp\{V_1(s), V_3(s)\}.$

Therefore

$U = \cos\varphi V_1(s) + \sin\varphi V_3(s).$

U is the linear combination of $V_1(s)$ and $V_3(s)$. By differentiating the equation $\langle V_2(s), U\rangle = 0$, we obtain

$$\langle -k_1 V_1(s) + k_2 V_3(s), U\rangle = 0,$$
$$-k_1(s)\langle V_1(s), U\rangle + k_2(s)\langle V_3(s), U\rangle = 0,$$
$$-k_1(s)\cos\phi + -k_2(s)\sin\phi = 0$$

By using the last equation, we see that

$H = const.$

$(\Leftarrow)$ Let $H(s)$ be constant for $\forall s \in I$, and $\lambda = \tan\phi$, then we obtain

$$U = \cos\phi V_1(s) + \sin\phi V_3(s)$$

If U is a constant vector, then we have

$$D_{\dot\alpha} U = \left(k_1(s)\cos\phi - \sin\phi k_2(s)\right)V_2(s)$$

By substituting $H(s) = \tan\varphi$ is in the last equation, we see that

$$k_1(s)\cos\phi - k_2 \sin\phi = 0$$

and so

$U = const.$

If α is an inclined curve with the slope axis $Sp\{U\}$, then

$$\langle \alpha'(s), U\rangle = \langle V_1(s), \cos\phi V_1(s) + \sin\phi V_3(s)\rangle$$
$$= \cos\phi\langle V_1(s), V_1(s)\rangle + \sin\phi\langle V_1(s), V_3(s)\rangle,$$

and we obtain

$$\langle \alpha'(s), U\rangle = \cos\varphi = const$$

([7]).

Definition: Let S^2 and α be a sphere in E^3 and a helix that lies on the sphere S^2. The curve α is called a spherical helix which lies on the sphere [12].

Definition: Let α be a helix in $M \subset E^3$. We define a surface element of M is the part of a tangent plane at the neighbour of the point of the helix that lie on M. Instead of the geometric plane of these surface elements along the helix α which lie sphere M is called a helix strip.

Definition: Let S^2 be a sphere and and α a helix which lie on S^2 in E^3. We define a surface element S^2 is the part of a tangent plane at the neighbour of the point of the helix that lie on S^2. The locus of these surface elements along the helix α which lie on the sphere S^2 is called spherical helix strip.

IV. FINDING ENERGY OF THE STRIP BY USING ITS CURVATURES

In this section we find energy of the strip by using its curvatures k_n, k_g and t_r.

We know that a strip has three curvatures $k_n = -b, k_g = c, t_r = a$ be the normal curvature, geodesic curvature, geodesic torsion of the curve-surface pair [1,2,3,5,6,8]. We can find the energy of the strip by using its two curvatures $k_n = -b, k_g = c,$

$$\varepsilon_n = \int k_n^2 ds$$

or

$$\varepsilon_g = \int k_g^2 ds$$

$(\alpha, M) \subset E^3$ is a strip, so we have its energies

$$\varepsilon_n = k_n^2 s + l$$

$$\varepsilon_n = -b^2 s + l$$

or

$$\varepsilon_g = k_g^2 s + l$$

$$\varepsilon_g = c^2 s + l\ .$$

Theorem 4.1. (Terquem Theorem) Let M_1 and M_2 be the different surfaces in E^3 and α be a curve but not a planar curve and β be a curve in M_2.

i. The points of the curves α ve β corresponds to each other 1:1 on a plane ε which rolls on the M_1 and M_2, such that the distance is constant between corresponding points.

ii. (α, M_1) is a curvature strip.

iii. (β, M_2) is a curvature strip.

Proof. Claim: Two of the three lemmas gives third ([10]). It is obviously from the Phd. thesis by Keles.

By applying the similar way in proof of the Theorem 3.1 in [10] to the strip of the spherical helix strip, we give the following theorem.

Theorem 4.2. (Joachimsthal Theorem) Let S^2 be a sphere and and M be a surface in E^3. Let the tangent planes of the surface M that along the curve β be the tangent planes of the sphere S^2 along the helix curve α at the same time. In this case, if we find the energy of the strip (β, M), the curve β is a helix, also a helix strip. If we find the energy of the

curve α on the spherical helix strip (S^2, M), we can find the energy of the curve β on (β, M) in type of the curvatures of the (β, M) and give a characterization.

Proof:

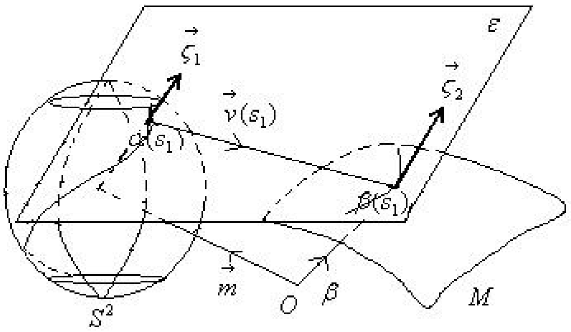

Figure 1.

Now Keles's proof help us to obtain the energy of the strip.

If the curve α is a helix on S^2, then it provides κ_1 / τ_1 is constant. We have to show that β is a helix strip on M, that is, $\frac{\kappa_2}{\tau_2}$ is constant.

By the Figure, we have

$$\beta(s_1) = \alpha(s_1) + \lambda(s_1)\vec{\nu}(s_1) \qquad (2)$$

where

$$\alpha(s_1) = \vec{m} + r\zeta_1(s_1) \qquad (3)$$

By differentiating both sides of (3), we see that

$$\vec{\xi}_1 = \frac{d\alpha_1}{ds_1} = r\frac{d\zeta_1}{ds_1}.$$

By (1),

$$\vec{\xi}_1 = r(b_1\xi_1 - a_1\eta_1),$$

We obtain $a_1 = 0$ and $b_1 = 1$.

r is the radius of the sphere. We denote $r = 1$. Since $\vec{m}$ is a position vector that goes to the center of the sphere, $\vec{m}$ is constant.

Since $\alpha_1 = 0$, (α, S^2) is a curvature strip. By the strips (α, S^2) and (β, M) are curvature strips and by the Terquem Theorem, we see that λ is non-zero constant. Let $\vec{\nu}(s_1)$ be a vector in $Sp\{\xi_1, \eta_1\}$, and let φ be the angle between $\vec{\xi}_1$ and $\vec{\nu}(s_1)$. Then we write

$$\vec{\nu}(s_1) = \cos\varphi\,\vec{\xi}_1 + \sin\varphi\,\vec{\eta} \qquad (4)$$

By substituting (3) and (4) in (2), and differentiating both sides, we obtain (5).

$$\frac{d\beta}{ds_1} = \frac{d\vec{m}}{ds_1} + \frac{d\vec{\zeta}_1}{ds_1} + \frac{d\lambda}{ds_1}(\cos\varphi\,\vec{\xi}_1 + \sin\vec{\eta}_1) + \lambda(s_1)\frac{d(\cos\varphi\,\vec{\xi}_1 + \sin\varphi\,\vec{\eta}_1)}{ds_1} \qquad (5)$$

Since the vector $\vec{m}$ and λ are constant, we obtain the following equation

$$\frac{d\beta}{ds_1} = \frac{d\vec{\zeta}_1}{ds_1} + \lambda(s_1)\frac{d(\cos\varphi\,\vec{\xi}_1 + \sin\varphi\,\vec{\eta}_1)}{ds_1}$$

or

$$\frac{d\beta}{ds_1} = \frac{d\vec{\zeta}_1}{ds_1} + \lambda(s_1)(-\frac{d\varphi}{ds_1}\sin\varphi\,\vec{\xi} + \cos\varphi\frac{d\xi_1}{ds_1}) + \frac{d\varphi}{ds_1}\cos\varphi\,\vec{\eta}_1 + \sin\varphi\frac{d\vec{\eta}_1}{ds_1}).$$

By (1), we obtain

$$\frac{d\beta}{ds_1} = \left[1 - \lambda(\frac{d\varphi}{ds_1} + c_1)\sin\phi\right]\vec{\xi}_1 + \lambda(\frac{d\phi}{ds_1} + c_1)\cos\phi\,\vec{\eta}_1 - \lambda\cos\phi\,\vec{\zeta}_1 \qquad (6)$$

Since the spherical helix and the surface M have the same tangent plane along the curves α and β, we can write

$$\left\langle \frac{d\beta}{ds_1}, \vec{\zeta}_1 \right\rangle = 0$$
.

By substituting (6) at the last equation, we obtain $\cos\varphi = 0$. By using that equation in (6), we have

$$\frac{d\beta}{ds_1} = (1 \pm \lambda c_1)\vec{\xi}_1 \qquad (7)$$

If we calculate the second and third derivatives of the curve β, then we get

$$\frac{d^2\beta}{ds_1^2} = \mp\lambda c_1'\,\vec{\xi}_1 + (1 \mp \lambda c_1)c_1\,\vec{\eta}_1 - (1 \mp \lambda c_1)\vec{\zeta}_1$$

$$\frac{d^3\beta}{ds_1^3} = \left[\mp\lambda c'' - (1 \mp \lambda c_1)c_1^2 - (1 \mp \lambda c_1)\right]\vec{\xi}_1 +$$

$$\left[\mp\lambda c_1 c_1' \mp \lambda c_1 c_1' + (1 \mp \lambda c_1)c_1'\right]\vec{\eta}_1 + (\mp\lambda c_1' \mp \lambda c_1')\vec{\zeta}_1$$

Since the same result is obtained by using other form of (7), we use the form $\frac{d\beta}{ds_1} = (1 - \lambda c_1)\vec{\xi}_1$ of (7) at the rest of our proof. By differentiating both sides of (7), we obtain

$$\frac{d\beta}{ds_1} = (1 - \lambda c_1)\vec{\xi}_1$$

$$\frac{d^2\beta}{ds^2_1} = -\lambda c_1' \vec{\xi}_1 + (1-\lambda c_1)c_1 \vec{\eta}_1 - (1-\lambda c_1)\vec{\zeta}_1$$

$$\frac{d^3\beta}{ds_1^3} = \left[-\lambda c'' - (1-\lambda c_1)c_1^2 - (1-\lambda c_1)\right]\vec{\xi}_1 + \left[3\lambda c_1 c_1' + c_1'\right]\vec{\eta}_1 + (2\lambda c_1')\vec{\zeta}_1$$

By applying Gram-Schmidt to the $\{\beta', \beta'', \beta'''\}$, then we have

$$F_1 = (1-\lambda c_1)\vec{\xi}_1$$

$$F_2 = (1-\lambda c_1)c_1 \vec{\eta}_1 - (1-\lambda c_1)\vec{\zeta}_1$$

$$F_3 = \frac{(1-\lambda c_1)c_1'}{c_1^2+1}\vec{\eta}_1 + \frac{(1-\lambda c_1)c_1' c_1}{c_1^2+1}\vec{\zeta}_1.$$

By [10], we have

$$\kappa_1^2 = b_1^2 + c_1^2,\ b_1 = 1 \tag{8}$$

and

$$\tau_1^2 = -a_1 + \frac{b_1' c_1 - b_1 c_1'}{b_1^2 + c_1^2}, a_1 = 0 \tag{9}$$

By (8) and (9), we see that

$$\tau_1 = \frac{-c_1'}{\kappa_1^2} \tag{10}$$

By using (10) in F_3, we obtain

$$F_3 = -(1-\lambda c_1)\tau_1 \vec{\eta}_1 - (1-\lambda c_1)\tau_1 \vec{\zeta}_1.$$

If we calculate κ_2 and τ_2, then we have

$$\kappa_2 = \frac{\kappa_1}{|1-\lambda c_1|}$$

and

$$\tau_2 = \frac{\tau_1}{|1-\lambda c_1|}$$

Dividing by κ_2 to τ_2, we obtain

$$\frac{\kappa_2}{\tau_2} = \frac{\kappa_1}{\tau_1} \tag{11}$$

$$\kappa_2 = \frac{\tau_2}{\tau_1}\kappa_1$$

$$\kappa_2 = \sqrt{b_1^2 + c_1^2}\ \frac{\dfrac{\tau_1}{|1-\lambda c_1|}}{\tau_1}$$

$$\kappa_2 = \frac{\sqrt{b_1^2 + c_1^2}}{|1-\lambda c_1|}$$

$$\kappa_2^2 = \frac{1+c_1^2}{(1-\lambda c_1)^2}$$

So we have

$$1 + c_1^2 = \kappa_2^2 (1-\lambda c_1)^2$$

We will use these equations for finding energy of the strip on Joachimsthal theorem by using its curvatures.

Corollary 4.3. Now we can find the energy of α on (S^2, M) and a relation between β on (β, M).

Let ε_1 the energy of the curve α. If we use the energy formulae and from (8), we calculate ε_1:

$$\varepsilon_1 = \int \kappa_1^2 ds$$
$$= \int \left(b_1^2 + c_1^2\right) ds$$
$$= \int \left(1 + c_1^2\right) ds$$
$$= \left(1 + c_1^2\right) s + l$$

Then we can find ε_1 in type of the curvature κ_2 of the curve β by using Terquem theorem and we have

$$\varepsilon_1 = \kappa_2^2 \left(1 - \lambda c_1^2\right) s + l$$

Let ε_2 the energy of the curve β by using its curvature κ_2 and the curvatures and b_2 and c_2 of the strip. In actually we should write ε_2:

$$\varepsilon_2 = \int \kappa_2^2 ds$$
$$= \int \left(b_2^2 + c_2^2\right) ds$$
$$= \left(b_2^2 + c_2^2\right) s + l$$

Also we obtain from the proof of the theorem ε_2,

$$\varepsilon_2 = \int \kappa_2^2 ds$$
$$= \kappa_2^2 s + l$$
$$= \int \frac{1+c_1^2}{(1-\lambda c_1)^2} s + l$$

we have

$$\kappa_2^2 = \frac{\varepsilon_1}{(1-\lambda c_1)^2 s} + l$$

So we obtain

$$\varepsilon_2 = \frac{\varepsilon_1}{(1-\lambda c_1)^2 s} s + l$$

$$\varepsilon_2 = \frac{\varepsilon_1}{(1-\lambda c_1)^2} + l$$

We obtain a characterization on finding energies of the strip and a curve.

V. Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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