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Comparison of Closed Repeated Newton-Cotes Quadrature Schemes with Half-Sweep Iteration Concept in Solving Linear Fredholm Integro-Differential Equations

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Abstract- The purpose of this paper is to apply half-sweep iteration concept with Gauss-Seidel (GS) iterative method namely Half-Sweep Gauss-Seidel (HSGS) method for solving high order closed repeated Newton-Cotes (CRNC) quadrature approximation equations associated with numerical solution of linear Fredholm integro-differential equations. Two different order of CRNC i.e. repeated Simpson's $\frac{1}{3}$ and repeated Simpson's $\frac{3}{8}$ schemes are considered in this research work. The formulation the implementation the proposed methods are explained. In addition, several numerical simulations and computational complexity analysis were carried out to authenticate the performance of the methods. The findings show that the HSGS iteration method is superior to the standard GS method. As well the high order CRNC quadrature schemes produced more precise approximation solution compared to repeated trapezoidal scheme.

Keywords- *Linear Fredholm integro-differential equations, Newton-Cotes Closed Quadrature, central difference, Half-Sweep Gauss-Seidel.*

I. INTRODUCTION

In this paper we focus on numerical solutions for first and second order Fredholm types of linear integro-differential equations. Generally, linear Fredholm integro-differential equations (LFIDEs) can be defined as follows

$$\left.\begin{array}{l}\left[D^n+\sum_{i=0}^{n-1}P_i(x)D^i\right]y(x)=g(x)+\int_a^b K(x,t)y(t)dt \\ \qquad x,t\in[a,b]\end{array}\right\} \quad (1)$$

with most general boundary condition,

$$\sum_{i=0}^{n-1}\left[r_{ji}D^i y(x)\Big|_{x=a}+r_{jn+i}D^i y(x)\Big|_{x=b}\right]=\alpha_j$$
$$j=0,\ldots,n-1$$

where $K(x,t)$, $g(x)$, $P(x)$ for $i=0,\ldots,n-1$ are known functions, $y(x)$ is the unknown function to be determined and $D^i y(x)$ denote the i^{th} derivative of $y(x)$ with respect to x.

The linear LFIDEs occur in multiple diversified physical phenomena such as physical biology and engineering problems. Therefore numerical treatment is preferred in order to diagnose and solve the problems. In many application areas, it is necessary to use the numerical approach to obtain an approximation solution for the (1) such as finite difference-Gauss [1] Taylor collocation [2], Lagrange interpolation [3] and Taylor polynomial [4] and rationalized Haar functions [5] Tau [6],. Subsequently, generated system of linear equation has been solved by using iterative methods such as Conjugate Gradient [7], GMRES [8]. Based on extension work from [9], in this paper, discretization scheme based on family of closed repeated Newton-Cotes (CRNC) quadrature namely repeated Simpson $\frac{1}{3}$ (RS1) and repeated Simpson's $\frac{3}{8}$ (RS2) along with finite difference schemes will be implemented to discretize (1). Then the generated linear system will be solved by using Half-Sweep Gauss-Seidel (HSGS) iterative method.

Actually, the HSGS represents combination of half-sweep iteration concept with standard Gauss-Seidel (GS) method. The standard GS method is also known as Full-Sweep Gauss Seidel (FSGS) method. The concept of the half-sweep iteration method has been introduced by Abdullah [10] via the Explicit Decoupled Group (EDG) iterative method to solve two-dimensional Poisson equation. Half-sweep iteration concept is also known as the complexity reduction approach [11]. Following that, the application of the half-sweep iteration concept with the iterative methods has been extensively studied by many researchers; see [12-14].

The rest of this work is organized as follows. In Section II, the derivation of the approximation equation is elaborated. In section III formulation of the FSGS and HSGS iterative methods are shown. Meanwhile, some numerical results are illustrated in Section IV to assert the effectiveness of the proposed methods and concluding remarks are given in Section V.

II. APPROXIMATION EQUATIONS

Figure 1: a) and b) show distribution of uniform node points for the full- and half-sweep cases respectively. The full- and half-sweep iteration concept will compute approximate values onto node points of type ● only until the convergence criterion is reached. Then other approximate solutions at the remaining points (points of the different type ○) can be computed using the direct method [10, 11, 12 and 14].

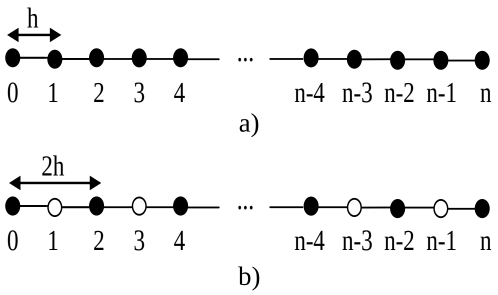

Figure 1. a) and b) show distribution of uniform node points for the full and half-sweep cases respectively

A. *Derivation of the Half-Sweep Qudrature Schemes*

Afore-mentioned, numerical approaches were used widely to solve LFIDEs than the analytical methods [15]. Therefore, CRNC quadrature schemes are applied to discretize the LFIDEs to form approximation of system of linear equations. Generally, quadrature formulas can be expressed as follows

$$\int_a^b y(t)dt = \sum_{j=0}^{n} A_j y(t_j) + \varepsilon_n(y) \tag{2}$$

where, $t_j \quad (j = 0,1,\ldots,n)$ are the abscissas of the partition points of the integration interval $[a,b]$. $A_j \quad (j = 0,1,\ldots,n)$ are numerical coefficients that do not depend on the function $y(t)$ and $\varepsilon_n(y)$ is the truncation error of (2). In formulating the full- and half-sweep approximation equations for (1), further discussion will be restricted onto quadrature methods, which is based on interpolation formulas with equally spaced data. Numerical coefficients A_j represented for following relation namely RT, RS1 and RS2 schemes respectively.

$$A_j = \begin{cases} \frac{1}{2} ph & j = 0, n \\ ph, & otherwise \end{cases} \tag{3}$$

$$A_j = \begin{cases} \frac{1}{3} ph, & j = 0, n \\ \frac{4}{3} ph, & j = p, 3p, 5p, \cdots, n - p \\ \frac{2}{3} ph, & otherwise \end{cases} \tag{4}$$

$$A_j = \begin{cases} \frac{3}{8} ph, & j = 0, n \\ \frac{3}{8} ph, & j = 3p, 6p, 9p \cdots, n - 3p \\ \frac{9}{8} ph, & otherwise \end{cases} \tag{5}$$

where the constants step size, h is defined as

$$h = \frac{b - a}{n} \tag{6}$$

n is the number of subintervals in the interval $[a, b]$ and then consider the discrete set of points be given as $x_i = a + ih$. The value of p which is corresponds to 1 and 2, represents the full- and half-sweep cases respectively.

B. *Derivation of the Half-Sweep Finite Difference Schemes*

In solving first order LFIDEs, differential part will be approximated by second order accuracy of first order derivative of finite difference scheme given by

$$y'(x_i) = \frac{y(x_{i+1}) - y(x_{i-1})}{2h} + O(h^2) \tag{7}$$

for $i = 1, 2, n-1$. However at the point x_n, second order accuracy of first derivative of backward difference, which is derived from the Taylor series expansion given as

$$y'(x_n) = \frac{3y(x_n) - 4y(x_{n-1}) + y(x_{n-2})}{2h} + O(h^2) \tag{8}$$

are considered. For solving second order LFIDEs, the second derivative central difference schemes can be derived as

$$y''(x_i) = \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} + O(h^2) \tag{9}$$

where h is size interval between nodes as mentioned in (6). For (7), (8) and (9) have the same order of the truncation error where mostly under our control because we can choose number of terms from the expansion of Taylor series. In order to obtain the finite grid work network for formulation of the full- and half-sweep central difference approximation equations over (1), the (2), (3) and (4) can be rewritten in general form as

$$y'(x_i) \cong \frac{y(x_{i+p}) - y(x_{i-p})}{2ph}, \tag{10}$$

for $i = 1p, 2p, 3p, \cdots, n-p$ and

$$y'(x_n) \cong \frac{3y(x_n) - 4y(x_{n-p}) + y(x_{n-2p})}{2ph}, \qquad (11)$$

for $i = n$

the second derivative of second order central difference schemes can be derived as

$$y''(x_i) = \frac{y(x_{i+p}) - 2y(x_i) + y(x_{i-p})}{(ph)^2} + O(h^2) \qquad (12)$$

for $i = 1p, 2p, 3p, \cdots, n-p$

where the value of p, which corresponds to 1 and 2, represents the full- and half -sweep respectively. In order to generate system of linear equation for first order LFIDEs, equations (2), (10) and (11) will be substituted into (1). The linear system generated either by the full-sweep or half-sweep approximation equation can be simply shown as

$$E \underset{\sim}{y} = \underset{\sim}{f} \,. \qquad (14)$$

where,

$$E = \begin{bmatrix} a_{p,p} & b_{p,2p} & d_{p,3p} & \cdots & d_{p,n-2p} & d_{p,n-p} & d_{p,n} \\ c_{2p,p} & a_{2p,2p} & b_{2p,3p} & \cdots & d_{2p,n-2p} & d_{2p,n-p} & d_{2p,n} \\ d_{3p,p} & c_{3p,2p} & a_{3p,3p} & \cdots & d_{3p,n-2p} & d_{3p,n-p} & d_{3p,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ d_{n-2p,p} & d_{n-2p,2p} & d_{n-2p,3p} & \cdots & a_{N-2p,n-2p} & b_{n-2p,n-p} & d_{n-2p,n} \\ d_{n-p,p} & d_{n-p,2p} & d_{n-p,3p} & \cdots & c_{n-p,n-2p} & a_{n-p,n-p} & b_{n-p,n} \\ d_{n,p} & d_{n,2p} & d_{n,3p} & \cdots & b_{n,2p} & e_{n,n-p} & \hbar_{n,n} \end{bmatrix}_{\left(\frac{n}{p}\right)\times\left(\frac{n}{p}\right)}$$

, Where,

$a_{i,i} = -2hP_i - 2hA_i K_{i,i}$,

$b_{i,j} = 1 - 2hA_j K_{i,j}$,

$c_{i,j} = -1 - 2hA_j K_{i,j}$,

$d_{i,j} = -2hA_j K_{i,j}$,

$e_{i,j} = -4 - 2hA_j K_{i,j}$,

$\hbar_{i,i} = -3 - 2hP_i - 2hA_i K_{i,i}$,

$$\underset{\sim}{f} = \begin{bmatrix} 2hg_p + (2hA_p K_{p,0} + 1)y_0 \\ 2hg_{2p} + (2hA_p K_{2p,0})y_0 \\ 2hg_{3p} + (2hA_p K_{3p,0})y_0 \\ \vdots \\ 2hg_{n-2p} + (2hA_p K_{n-2p,0})y_0 \\ 2hg_{n-p} + (2hA_p K_{n-p,0})y_0 \\ 2hg_N + (2hA_p K_{n,0})y_0 \end{bmatrix} \quad \text{and} \quad \underset{\sim}{y_n} = \begin{bmatrix} y_n(x_p) \\ y_n(x_{2p}) \\ y_n(x_{3p}) \\ \vdots \\ y_n(x_{n-2p}) \\ y_n(x_{n-p}) \\ y_n(x_n) \end{bmatrix}$$

where E is a dense coefficient matrix, $\underset{\sim}{f}$ is given function and $\underset{\sim}{y}$ is unknown function to be determined. Nevertheless, in solving first order LFIDEs, the combination of discretization schemes of such as central difference and repeated trapezoidal (CD-RT), central difference and repeated Simpson $\frac{1}{3}$ (CD-RS) and central difference and repeated Simpson $\frac{3}{8}$ (CD-RS2) leads to the non-positive definite coefficient matrices. Therefore, for GS iterative methods, the generated linear systems will be modified by multiplying the coefficient matrices with its transpose in order to strengthen the diagonal elements. The new linear system (14) can be simplified as

$$E^* \underset{\sim}{y} = \underset{\sim}{f}^* \qquad (15)$$

where,

$$E^* = E^T E$$

and

$$\underset{\sim}{f}^* = E^T \underset{\sim}{f}$$

Now the linear system (15) can be solved iteratively via FSGS and HSGS iterative methods.

For, second order LFIDEs, equations (2) and (11) will be substituted into (1) to generate linear system either by the full-sweep or half-sweep approximation equation can be simply shown as

$$G \underset{\sim}{y_n} = \underset{\sim}{\ell} \qquad (16)$$

where

$$G = \begin{bmatrix} \sigma_{p,p} & \varsigma_{p,2p} & \tau_{p,3p} & \cdots & \tau_{p,n-3p} & \tau_{p,n-2p} & \tau_{p,n-p} \\ \varsigma_{2p,p} & \sigma_{2p,2p} & \varsigma_{2p,3p} & \cdots & \tau_{2p,n-3p} & \tau_{2p,n-2p} & \tau_{2p,n-p} \\ \tau_{3p,p} & \varsigma_{3p,2p} & \sigma_{3p,3p} & \cdots & \tau_{3p,n-3p} & \tau_{3p,n-2p} & \tau_{3p,n-p} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \tau_{n-3p,p} & \tau_{n-3p,2p} & \tau_{n-3p,3p} & \cdots & \sigma_{n-3p,n-3p} & \varsigma_{n-3p,n-2p} & \tau_{n-2p,n-p} \\ \tau_{n-2p,p} & \tau_{n-2p,2p} & \tau_{n-2p,3p} & \cdots & \varsigma_{n-2p,n-3p} & \sigma_{n-2p,n-2p} & \varsigma_{n-p,n} \\ \tau_{n-p,p} & \tau_{n-p,2p} & \tau_{n-p,3p} & \cdots & \tau_{n-p,n-3p} & \varsigma_{n-p,n-2p} & \sigma_{n-p,n-p} \end{bmatrix}_{\left(\frac{n}{p}-1\right)\times\left(\frac{n}{p}-1\right)},$$

Where,

$\sigma_{i,i} = -2 - h^2 P_i - h^2 A_i K_{i,i}$,

$\varsigma_{i,j} = 1 - h^2 A_j K_{i,j}$,

$\tau_{i,j} = -h^2 A_j K_{i,j}$,

$$\underset{\sim}{\ell} = \begin{bmatrix} h^2 g_p + \left(1 + h^2 A_p K_{p,0}\right) y_0 + \left(h^2 A_n K_{p,n}\right) y_n \\ h^2 g_{2p} + \left(h^2 A_p K_{2p,0}\right) y_0 + \left(h^2 A_n K_{2p,n}\right) y_n \\ h^2 g_{3p} + \left(h^2 A_p K_{3p,0}\right) y_0 + \left(h^2 A_n K_{3p,n}\right) y_n \\ \vdots \\ h^2 g_{n-3p} + \left(h^2 A_p K_{n-3,0}\right) y_0 + \left(h^2 A_n K_{n-3p,n}\right) y_n \\ h^2 g_{n-2p} + \left(h^2 A_p K_{n-2,0}\right) y_0 + \left(h^2 A_n K_{n-2p,0}\right) y_n \\ h^2 g_{n-p} + \left(h^2 A_p K_{n-p,0}\right) y_0 + \left(-1 + h^2 A_n K_{n-p,0}\right) y_n \end{bmatrix}, \quad \text{and} \quad \underset{\sim}{y_n} = \begin{bmatrix} y_n(x_p) \\ y_n(x_{2p}) \\ y_n(x_{3p}) \\ \vdots \\ y_n(x_{n-2p}) \\ y_n(x_{n-p}) \\ y_n(x_n) \end{bmatrix}$$

where G is a positive definite, non symmetric coefficient matrix, $\underset{\sim}{f}$ is given function. and $\underset{\sim}{y}$ is unknown function to be determined.

III. Formulation of FSGS and HSGS Iterative Methods

In this section, generated system of linear equation of first order and second order LFIDEs will be solved by using FSGS and HSGS iterative methods as shown in (15) and (16). For first order LFIDEs, let the coefficient matrix, E^* be decomposed into

$$E^* = D - L - U \qquad (15)$$

where D, $-L$ and $-U$ are diagonal, strictly lower triangular and strictly upper triangular matrices respectively. In fact, the both iterative methods attempt to find a solution to the system of linear equations by repeatedly solving the linear system using approximations to the vector $\underset{\sim}{y}$. Iterations for both methods continue until the solution is within a predetermined acceptable bound on the error. By determining values of matrices D, $-L$ and $-U$ as stated in (15), the general algorithm for FSGS and HSGS iterative methods to solve (1) would be generally described in Algorithm 1.

Algorithm 1: FSGS and HSGS algorithms

(i) Initializing all the parameters. Set $k = 0$.
1.

(ii) F $i = 1p, 2p, \cdots, n-p$ and $j = 1, p, 2p, \cdots, n-p, n$, calculate

$$\underset{\sim}{y}_i^{(k+1)} = \frac{1}{E_{i,i}^*}\left(f_i^* - \sum_{j=p,2p}^{i-p} E_{i,j}^* \underset{\sim}{y}_j^{(k+1)} - \sum_{j=i+p,i+2p}^{n} E_{i,j}^* \underset{\sim}{y}_j^{(k)} \right)$$

(iii) Convergence test.
If there error of tolerance $\left\| y_i^{(k+1)} - y_i^{(k)} \right\| \le \varepsilon = 10^{-10}$ is satisfied, then algorithms stop.

(iv) Else, set $k = k+1$ and go to step (ii).

For second order LFIDEs, the general algorithm for FSGS and HSGS iterative methods to solve (1) would be generally described in Algorithm 2.

$$G = D - L - U \qquad (15)$$

where D, $-L$ and $-U$ are diagonal, strictly lower triangular and strictly upper triangular matrices respectively.

Algorithm 2: FSGS and HSGS algorithms

(i) Initializing all the parameters. Set $k=0$.
1.

(ii) F $i=1p,2p,\cdots,n-p$ and $j=1,p,2p,\cdots,n-p$, calculate

$$\underset{\sim i}{y}^{(k+1)}=\frac{1}{G_{i,i}}\left(\ell_i-\sum_{j=p,2p}^{i-p}G_{i,j}\underset{\sim j}{y}^{(k+1)}-\sum_{j=i+p,i+2p}^{n-1}G_{i,j}\underset{\sim j}{y}^{(k)}\right)$$

(iii) Convergence test.
If there error of tolerance $\left\|y_i^{(k+1)}-y_i^{(k)}\right\|\le\varepsilon=10^{-10}$ is satisfied, then algorithms stop.

(iv) Else, set $k=k+1$ and go to step (ii).

IV. Numerical Simulations

In order to evaluate the performances of the HSGS iterative methods described in the previous section, several numerical experiments were carried out. In this paper, we will only consider well posed equations and the case where $a=0$ and $b=1$.

Problem 1 [16]. Consider the first order LFIDE

$$y'(x)=1-\frac{1}{3}x+\int_0^1 xy(t)dt\,,\quad 0\le x\le 1 \qquad (16)$$

with bondary condition

$$y(0)=0$$

and exact solution is

$$y(x)=x\,.$$

Problem 2 [17]. Consider the second order LFIDE

$$y''(x)=x-2+\int_0^1 60(x-t)y(t)\,dt\,,0\le x\le 1 \qquad (17)$$

with bondary conditions

$$y(0)=0 \text{ and } y(1)=0$$

with exact solution given as

$$y(x)=x\,.$$

There are three parameters considered in numerical comparison such as number of iterations, execution time and maximum absolute error. As comparisons, the Standard or Full Sweep Gauss-Seidel (FSGS) method acts as the control of comparison of numerical results. Throughout the simulations, the convergence test considered the tolerance error of the $\varepsilon=10^{-10}$.

V. Conclusions

In this work, we have implemented half-sweep iterative method on high order closed composite Newton Cotes quadrature schemes to solve LFIDEs. Based on Table III and Table IV, the half-sweep iteration concept on quadrature and central difference schemes with GS iterative method have decreased the number of iterations and execution time approximately 62.81%-74.23% and 85.56%-96.93% respectively for Problem 1 and 73.21%-76.25% and 46.71%-83.05% respectively for problem 2. Based on Table 1 and Table 2 the accuracy of numerical solutions for CD-RS1 and CD-RS2 schemes are more accurate than the CD-RT scheme. Overall, the numerical results have shown that the HSGS method is more superior in term of number of iterations and the execution time than standard method.

TABLE I. Number of arithmetic operations per iterations involved in a node point based on FSGS and HSGS method for First Order Linear FIDE

	Arithmetic Operations Per Node	
	ADD/SUB	**MUL/DIV**
FSGS	$n(n-1)$	$n(n+1)$
HSGS	$\frac{n}{2}\left(\frac{n}{2}-1\right)$	$\frac{n}{2}\left(\frac{n}{2}+1\right)$

TABLE II. Number of arithmetic operations per iterations involved in a node point based on FSGS and HSGS method for Second Order Linear FIDE

	Arithmetic Operations Per Node	
	ADD/SUB	**MUL/DIV**
FSGS	$(n-1)^2$	n^2-1
HSGS	$\left(\frac{n}{2}-1\right)^2$	$\frac{n^2}{4}-1$

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TABLE III. COMPARISON OF A NUMBER OF ITERATIONS, EXECUTION TIME (SECONDS) AND MAXIMUM ABSOLUTE ERROR FOR THITERATIVE METHODS USING CD-RT, CD-RS1 AND CD-RS2 DISCRETIZATION SCHEMES FOR PROBLEM 1.

Mesh Size	Schemes & Methods	Number of iteration		Execution time		Maximum absolute error	
		FSGS	HSGS	FSGS	HSGS	FSGS	HSGS
24	CD-RT	7814	2907	5.93	0.89	1.653E-4	6.620E-4
	CD-RS1	7964	2962	6.53	0.96	4.767E-8	1.654E-8
	CD-RS2	7811	2903	6.47	1.01	1.621E-8	6.021E-8
48	CD-RT	23006	7814	108.77	7.40	4.119E-5	1.653E-4
	CD-RS1	23428	7964	120.52	6.77	1.518E-8	4.767E-8
	CD-RS2	23004	7811	116.62	8.01	1.489E-8	4.670E-8
72	CD-RT	45002	14536	684.15	40.01	1.807E-5	4.119E-5
	CD-RS1	45756	14810	730.15	35.73	3.122E-8	9.269E-8
	CD-RS2	45001	14534	723.21	44.65	3.065E-8	9.089E-8
96	CD-RT	73430	23006	2469.69	142.92	9.828E-6	4.119E-5
	CD-RS1	74614	23428	2753.81	124.61	5.291E-8	1.518E-8
	CD-RS2	73428	23004	2631.58	154.56	5.199E-8	1.488E-8
120	CD-RT	107988	33174	10347.03	429.21	3.506E-6	2.623E-5
	CD-RS1	109685	33759	10460.84	328.75	1.233E-8	2.249E-8
	CD-RS2	107987	10950	11571.30	319.97	7.888E-8	6.708E-8

TABLE IV. COMPARISON OF A NUMBER OF ITERATIONS, EXECUTION TIME (SECONDS) AND MAXIMUM ABSOLUTE ERROR FOR THITERATIVE METHODS USING CD-RT, CD-RS1 AND CD-RS2 DISCRETIZATION SCHEMES FOR PROBLEM 2.

Mesh Size	**Schemes & Methods**	**Number of iteration**		**Execution time**		**Maximum absolute error**	
		FSGS	HSGS	FSGS	HSGS	FSGS	HSGS
24	CD-RT	502	130	0.22	0.05	4.656E-4	2.885E-3
	CD-RS1	497	134	0.29	0.08	2.414E-6	3.246E-6
	CD-RS2	497	134	0.31	0.09	5.435E-6	3.235E-6
48	CD-RT	2101	502	0.49	0.26	1.164E-4	7.912E-4
	CD-RS1	2097	497	0.50	0.34	1.389E-8	2.414E-7
	CD-RS2	2097	497	0.57	0.28	3.279E-8	1.266E-8
72	CD-RT	4628	1183	1.17	0.34	5.172E-5	3.627E-4
	CD-RS1	4625	1179	1.18	0.36	8.892E-8	3.777E-8
	CD-RS2	4625	1179	1.20	0.45	3.835E-8	3.881E-8
96	CD-RT	8034	2101	2.30	0.49	2.905E-5	2.072E-4
	CD-RS1	8032	2097	2.34	0.50	4.345E-8	2.136E-8
	CD-RS2	8032	2097	2.69	0.61	3.168E-8	2.136E-8
120	CD-RT	12278	3251	4.37	0.72	1.854E-5	1.338E-4
	CD-RS1	12276	3249	4.10	0.81	7.967E-8	1.371E-8
	CD-RS2	12276	3249	4.77	0.86	7.484E-8	1.371E-8