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On Modeling Tailings Deposition: Analytical and Numerical Methods

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Abstract- One of the main goals of Science is to model nature's behavior by means of mathematical equations. Such equations tend to ratify some of the basic notions one has about a given phenomena. Conservation of mass, continuity of fluids and equilibrium of phases are some of these "intrinsic" properties which are related to the physical interpretation of a given phenomenon. This way, while considering tailings deposition, the latter must satisfy field equations regarding conservation of the solid phase besides continuity and equilibrium of the liquid phase. In the present paper, a rigorous mathematical approach to the modeling of the main characteristics of tailings is presented. In short, an analytical solution deduced by means of Laplace transform is compared to numerical solutions based on finite differences method, Lax-Wendroff method and the Cubically Interpolated Pseudo-particle (CIP) method. It is shown that CIP method overcomes the problem of spurious numerical dissipation induced by the other numerical methods analyzed. Also, a study of case is done and the numerical solution closely matches the observed experimental data.

Keywords- *Tailings, Hydraullic Deposition, Finite Difference Method, Lax-Wendroff Method, Cubically Interpolated Pseudo-particle.*

I. INTRODUCTION

In [1] it has been presented a model to simulate the bed load transport of heterogeneous sediments. The model couples Navier-Stokes equations, representing continuity and equilibrium of the fluid phase, with the equation for the conservation of mass of the solid sediments. Besides, a constitutive relation that describes the rate of transported sediments should also be adopted.

Heterogeneous sediments or tailings, comprising iron and quartz particles, are commonly produced during the extraction and concentration of iron. The bed load transport phenomenon, in which the particles roll when carried by a thin film of water, is the most representative of the actual condition that is observed during the construction of tailings dams using hydraulic deposition techniques. Despite some engineering drawbacks, mainly if the upstream construction method is used, this is the solution for waste disposal generally adopted by most mining industries, due to its relatively low costs.

The coupled model for the conditions of equilibrium and continuity of the fluid phase, plus the continuity of the solid sediments, results in a system of hyperbolic equations. In this kind of problem the material is transported without dissipation. Nevertheless, if this system of equation is solved using classic techniques, such as the explicit Finite Differences Method (FDM), the solution introduces spurious dissipations that are purely numerical without any relation with the real physical problem. In order to overcome this problem, another technique known as the Cubic Interpolated Pseudo-particle method (CIP) is adopted in this paper.

II. COUPLED MODEL FOR BED LOAD TRANSPORT

In this section, a mathematical model for the bed load transport of heterogeneous sediments is presented. The model couples the behavior of the fluid phase and the sediments, composed of particles of quartz and iron. The objective when developing this model was to devise a mathematical and numerical tool that could help to forecast the profile of hydraulic deposition, including segregation, in tailings dams. Parametric analyses using this model, are very useful to gain further insight about the mechanisms involved during the deposition process, such as the distribution of density, porosity and grain sizes along the profile, as a function of the production variables (discharge and pulp concentration).

Considering the continuity of the fluid phase, equilibrium in the fluid, continuity of transported sediments and a relation for the rate of sediment transport, respectively, the following equations were deduced [1]:

$$a_t + a \cdot u_x + u \cdot a_x = 0 \tag{1}$$

$$u_t + u \cdot u_x + g \cdot \left(a_x + z_{bx}\right) = -\frac{g \cdot u^2}{C_h^2 \cdot a} \tag{2}$$

$$z_{bt} + s_x = 0 \tag{3}$$

$$s = \sum_k m_k u^{n_K} \tag{4}$$

in which u is the velocity of transport of sediments, a is the film of water above the bed of the deposition profile, z_b is the height of deposited sediments, s is the rate of transported sediments composed of particles of two types k (quartz, Qz and iron, Fe), C_h is the coefficient of Chèzy, m_k and n_k are empirical constants dependent on the properties of the sediments, x is the horizontal distance, as illustrated in Error! Reference source not found., and t is time.

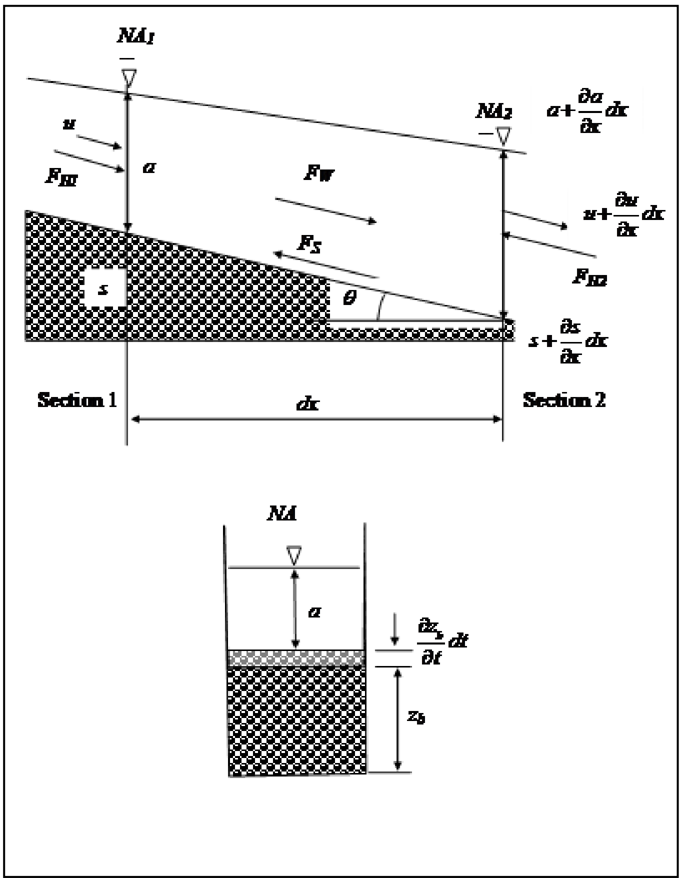

Figure 1. Details of an infinitesimal element in the upstream slope

In [2] it has been presented a numerical solution for the system (1)-(4), based on the Finite Differences Method using advanced differences in time and central differences in the space domain. The resulting explicit algorithm is summarized as follows:

$$u_k^{n+1} = u_k^n - \frac{g\Delta t}{C_h^2}\frac{\left(u_k^n\right)^2}{a_k^n} - \frac{\Delta t}{2\Delta x}\left\{u_k^n\left[u_{k+1}^n - u_{k-1}^n\right] + \right. \\ \left. +g\left(\left[a_{k+1}^n - a_{k-1}^n\right] + \left[z_{bk+1}^{\ n} - z_{bk-1}^{\ n}\right]\right)\right\} \tag{5}$$

$$a_k^{n+1} = a_k^n - \frac{\Delta t}{2\Delta x}\left\{a_k^n\left[u_{k+1}^n - u_{k-1}^n\right] + \right. \\ \left. +u_k^n\left[a_{k+1}^n - a_{k-1}^n\right]\right\} \tag{6}$$

$$z_{bk}^{\ n+1} = z_{bk}^{\ n} - \frac{\Delta t}{2\Delta x}\left[s_{k+1}^n - s_{k-1}^n\right] \tag{7}$$

$$s_k^{n+1} = m_{Qz}\left[u_k^n\right]^{n_{Qz}} + m_{Fe}\left[u_k^n\right]^{n_{Fe}} \tag{8}$$

It has been compared in [2] the solution obtained with the Finite Differences algorithms in (5)-(8) with the experimental data obtained in [3] from laboratory hydraulic deposition tests. The numerical and empirical results were also compared with analytical solutions obtained by [4] for simplified boundary conditions, expressed as follows:

$$z_b\left(x,t\right) = A\cdot\left\{\sqrt{Bt}\cdot\exp\left(-E\frac{x^2}{t}\right) - D\cdot x\cdot\text{erfc}\left(\sqrt{E}\frac{x}{t}\right)\right\} \tag{9}$$

in which:

$$A = \frac{z_b\left(0,t\right)}{\sqrt{Bt}}; B = \frac{60\cdot s_0\cdot i_0}{\pi\cdot a_0}; D = \frac{3\cdot i_0}{a_0}; E = \frac{9\cdot i_0}{60\cdot a_0\cdot s_0} \tag{10}$$

$$\text{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3\cdot 1!} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots\right) \tag{11}$$

a_0, s_0 and i_0 are the initial values for the film of water, solid transport rate and beach bed inclination, respectively.

When comparing the analytical solution and the FDM solution in (5)-(8), the numerical results were satisfactory only for a time-space discretization with a Courant number (C=u_0.Δt/Δx) equal to one. However, for C values less than unit the numerical solutions showed spurious dissipation and the results did not converge to those obtained analytically and empirically. This numerical error is investigated in [5] and in the present paper. The solution to such numerical problem may be achieved using the so-called Cubic Interpolated Pseudo-particle (CIP) method (discussed in [6]).

III. CIP METHOD

The cubic interpolated pseudo-particle method, CIP, as proposed in [7], is used here to find an approximate solution $u(x, t)$ to the advection problem. For a given constant advection velocity $v > 0$, the solution should propagate the information, displacing the curve by an amount equal to $\Delta x = v\Delta t$ to the right during the time interval Δt. For this to happen, the following hypothesis is assumed as illustrated in Figure 2 (a):

$$u(x,t) \cong u\left(x - \Delta x, t - \Delta t\right) \tag{12}$$

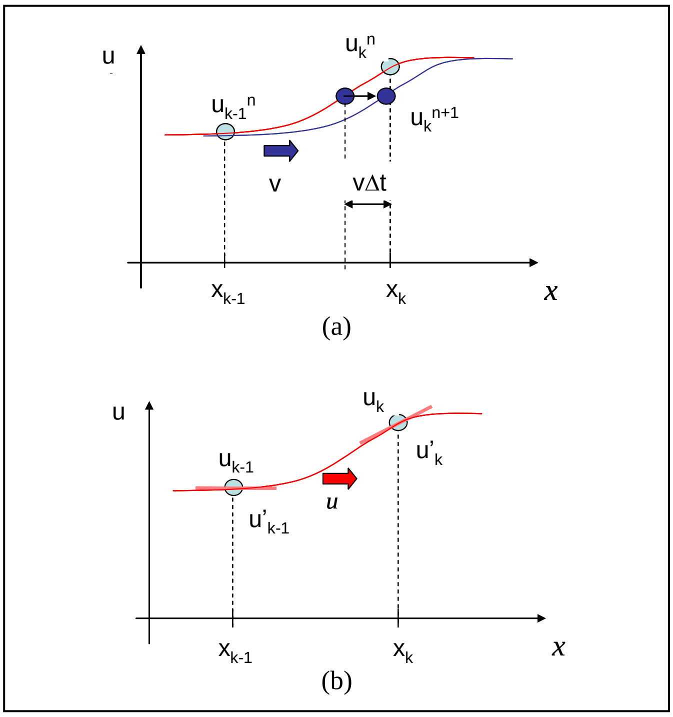

Figure 2. Hypothesis of CIP Method (modified from Moriguchi, 2005).

In the CIP method [8], the discrete solution u_k^n at time n for a mesh of points x_k in the space domain x is smoothed by approximating a Hermite cubic polynomial $U(x)$ in each space interval of length Δx between successive points $[x_{k-1}, x_k]$. The general form of the polynomial is given by:

$$U(x) = a_{k-1}^n \left(x - x_{k-1}^n\right)^3 + b_{k-1}^n \left(x - x_{k-1}^n\right)^2 + \\ + c_{k-1}^n \left(x - x_{k-1}^n\right) + d_{k-1}^n \tag{13}$$

The space derivatives of the cubic Hermite polynomial are given by:

$$U'(x) = 3a_{k-1}^n \left(x - x_{k-1}^n\right)^2 + 2b_{k-1}^n \left(x - x_{k-1}^n\right) + c_{k-1}^n \tag{14}$$

The CIP method forces the polynomial approximation and its derivatives, $U(x)$ and $U'(x)$, to match the discrete values, $u(x, t)$ and $u'(x, t)$, at the extremes of each space interval $[x_{k-1}, x_k]$ (see Figure 2b):

$$u(x_{k-1}, t_k) = U\left(x_{k-1}^n\right) = u_{k-1}^n; \\ u'(x_{k-1}, t_k) = U'\left(x_{k-1}^n\right) = u_{k-1}'^n \tag{15a}$$

$$u(x_k, t_k) = U\left(x_k^n\right) = u_k^n; \\ u'(x_k, t_k) = U'\left(x_k^n\right) = u'^n_k \tag{15b}$$

From the hypothesis presented in (15a) and using (13) and (14), it is possible to determine the coefficients c_{k-1} and d_{k-1}:

$$c_{k-1}^n = u_{k-1}'^n; \quad d_{k-1}^n = u_{k-1}^n \tag{16}$$

Noticing that $\Delta x = x_k^n - x_{k-1}^n = v\Delta t$ and applying the hypothesis presented in (15b) to the extreme values in (7) and (8), it is possible to determine the other coefficients a_{k-1}^n and b_{k-1}^n as:

$$a_{k-1}^n = \frac{\left(u_{k-1}'^n + u_k'^n\right)}{\Delta x^2} + \frac{2\left(u_{k-1}^n - u_k^n\right)}{\Delta x^3} \tag{17}$$

$$b_{k-1} = \frac{3\left(u_k^n - u_{k-1}^n\right)}{\Delta x^2} - \frac{\left(2u_{k-1}'^n + u_k'^n\right)}{\Delta x} \tag{18}$$

Now that all constants are determined, the discrete values of $u(x, t)$ and $u'(x, t)$ may be propagated to the next time step n+1 as follows:

$$u_k^{n+1} = a_{k-1}^n (v\Delta t)^3 + b_{k-1}^n (v\Delta t)^2 + u_{k-1}'^n (v\Delta t) + u_{k-1}^n \tag{19}$$

$$u_k'^{n+1} = 3a_{k-1}^n (v\Delta t)^2 + 2b_{k-1}^n (v\Delta t) + u_{k-1}'^n \tag{20}$$

The operations previously explained should be performed for all points of the mesh, for each interval in the space domain. This allows to estimate values of u and u' explicitly at the next time step n+1, given these values at time step n. Therefore the initial solution at t=0 is propagated in time during as many time steps as necessary. Notice, however, that the CIP scheme requires not only the values of the function at all space points as an initial condition u_k^0, but also the values of the derivatives, $u_k'^0$. If these derivatives are not explicitly given as a continuous function, $u'(x, 0) = f'(x)$, it is still possible to estimate them using the central finite differences of the initial discrete function values [9]:

$$u_k'^n = \frac{u_{k+1}^n - u_{k-1}^n}{2\Delta x} \tag{21}$$

The CIP method can eliminate the dissipation and oscillation problems observed when solving the advection equation with the Finite Differences and Lax-Wendroff methods with Courant numbers less than unit. This is fundamental for an independent and efficient discretization of the problem in both time and space domain.

The general mathematical formulation of an advective problem, in one-dimensional space, is described by the following hyperbolic equation:

$$u_t + v \cdot u_x = 0 \tag{22}$$

in which $v > 0$ is the advection velocity and $v.u_x$ is the so called advective term. The independent variable u is function both of space x and time t; u_x and u_t denote its derivatives.

One may notice that (22) represents the transport of u along axis x towards the right-hand side when v > 0. Since

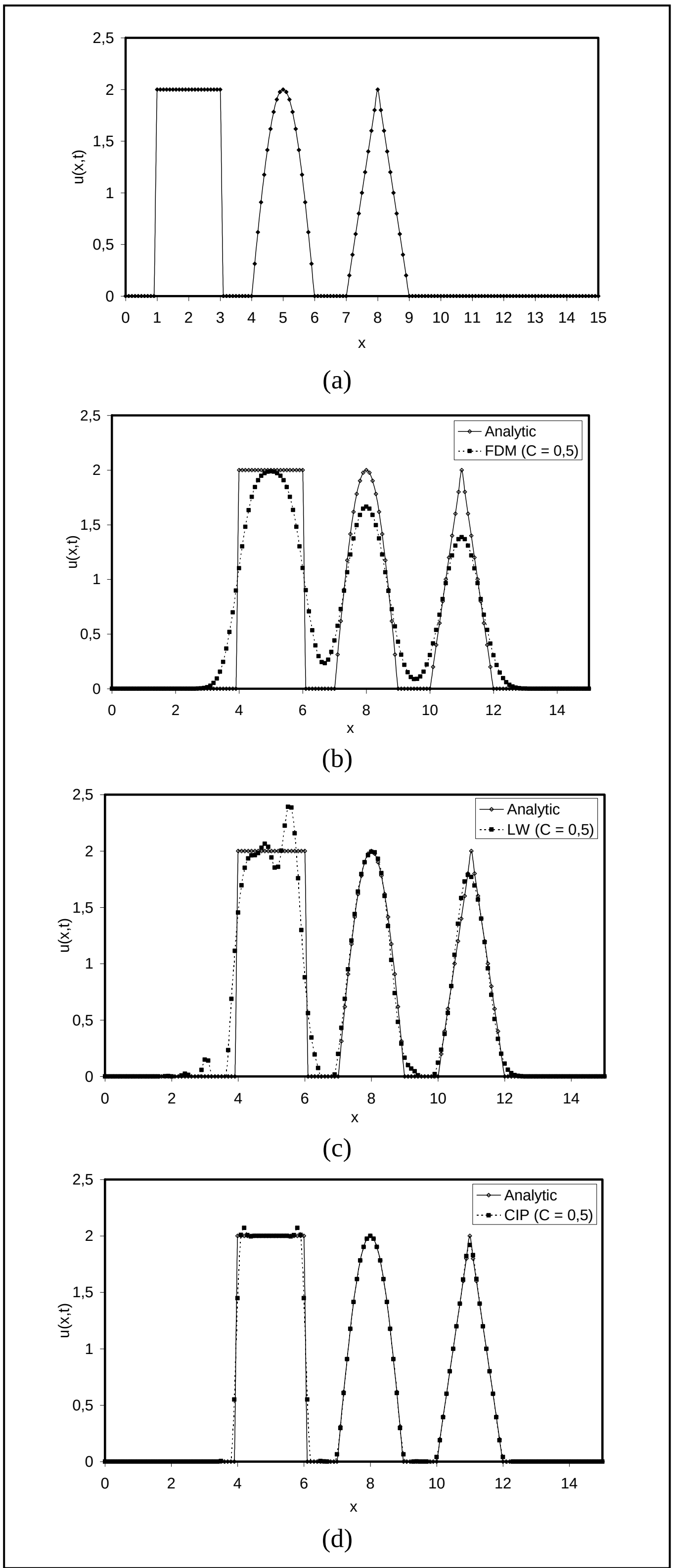

Figure 3. Solution for the Advection Problem with (a) particular initial condition, using (b) FDM, (c) Lax-Wendroff and (d) CIP methods and a Courant number C=0.5.

this equation does not contain a dissipative term, u_{xx}, then the value of u should just be transported along x, without undergoing any alteration, between time t0 and t0 + Δt. The absence of dissipative phenomena implies that any discontinuity in the initial conditions should propagate to the solution at any time t > 0. This implies that hyperbolic equation admits discontinuous solutions, and the numerical method adopted to solve these equations should be able to deal with such discontinuities efficiently.

The initial value problem, or Cauchy problem, for the advection equation consists in finding a function u(x, t) in the semi-space $D = \{(x,t)/t \geq 0, -\infty < x < \infty\}$, that satisfies both (22) and a particular initial condition. In general the solution is continuous and sufficiently differentiable, but this is not always the case.

As an example, consider the phenomenon of a wave propagating to the right hand side with the particular initial condition described in Figure 3(a).

In Figure 3, the solution for the advection problem, was obtained using the Finite Differences, Lax-Wendroff and CIP methods for a Courant number C=0.5. These solutions are shown for time equal to 3 s. Notice that the initial solution propagated in time without any loss of information, dissipation or oscillations for the CIP method.

Figure 4 shows the absolute error compared with the analytical solution.

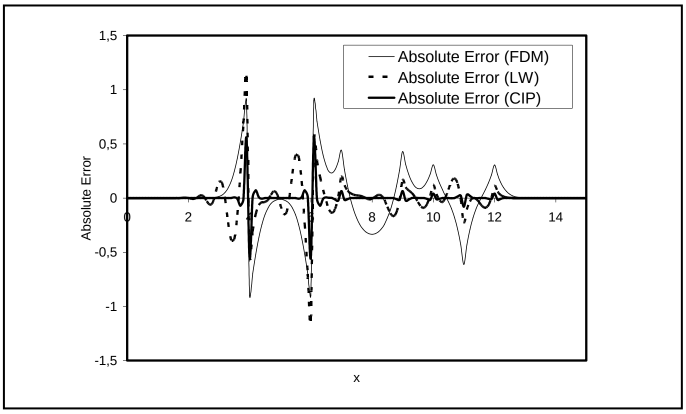

Figure 4. Absolute Error

IV. Coupled Bed Load Transport Problem Solved By CIP

The CIP method is efficient to solve advection problems, which results in hyperbolic equations for which dissipation phenomena are not present or may be disregarded. In order to use the CIP method, according to an algorithm known as particle-in-cell [10], the transport model in (1)-(4), must be split in two parts: an advective (Lagrangian) phase, and a non-advective (Eulerian) phase.

The non-advective equations governing the model are the following:

$$a_t = -a \cdot u_x \tag{23}$$

$$u_t = -\frac{g \cdot u^2}{C^2 \cdot a} - g\left(a_x + z_{bx}\right) \tag{24}$$

$$z_{bt} + s_x = 0 \tag{25}$$

$$s = \sum_k m_k u^{n_k} \tag{26}$$

The advective equations of the problem are as follows:

$$a_t + u \cdot a_x = 0 \tag{27}$$

$$u_t + u \cdot u_x = 0 \tag{28}$$

Therefore the equation governing continuity of the fluid phase, (1), was divided into a non-advective part in (23) and an advective part in (27). The same process was applied to the equation governing equilibrium of the fluid phase, (2), which was split into (24) and (28).

The CIP method is applied only to the advective equations, thus avoiding the introduction of spurious numerical dissipations in the solution. The non-advective equations may be solved using the Finite Differences Method (FDM) or any other method. In this case the correspondent non-advective formulations (23)-(26), may be re-written as:

$$a_k^{n+1} = a_k^n - \frac{\Delta t}{2 \cdot \Delta x}\left\{a_k^n \cdot \left(u_{k+1}^n - u_{k-1}^n\right)\right\} \tag{29}$$

$$u_k^{n+1} = u_k^n - \frac{g\Delta t}{C^2}\frac{\left(u_k^n\right)^2}{a_k^n} - $$

$$-\frac{g\Delta t}{2\Delta x}\left\{\left(a_{k+1}^n - a_{k-1}^n\right) + \left(z_{bk+1}^{\ n} - z_{bk-1}^{\ n}\right)\right\} \tag{30}$$

$$z_{bk}^{\ n+1} = z_{bk}^{\ n} - \frac{\Delta t}{2\Delta x}\left[s_{k+1}^n - s_{k-1}^n\right] \tag{31}$$

$$s_k^{n+1} = m_{Qz}\left[u_k^n\right]^{n_{Qz}} + m_{Fe}\left[u_k^n\right]^{n_{Fe}} \tag{32}$$

After solving the non-advective equations, these solutions are used as initial conditions for the advective equations. The advective phase, (27) and (28), is solved using the CIP method as described by the authors in this paper and in [5].

V. EXPERIMENTAL SIMULATION AND MODEL VALIDATION

Considering the importance of predicting the behaviour of hydraulic fill structures in the field and afterwards analyzing the performance of different kinds of laboratory simulation tests, a hydraulic deposition simulation apparatus was developed at the University of Brasilia [3]. The apparatus consists of a depositional channel, 6.0 m long, 0.4 m wide and 1.0 m high. The channel was built using steel profiles and panels of tempered glass. This kind of wall permits the observation of the evolution of the deposition process during the entire test. Figure 5 shows a general view of hydraulic deposition simulation tests (HDST), developed at the University of Brasilia.

Figure 5. General view of the HDST equipment developed at the University of Brasilia

A series of comparisons were made between the HDST results obtained by [3] and those forecasted by the coupled model solved by analytical, FDM and CIP techniques, for different Courant numbers. According to [3], the main characteristics of the material used in the HDST are those presented in Tables 1 and 2. In Tables 1 and 2, Fe is the percentage of iron particles, Cw is the concentration of solid particles (quartz and iron) in the slurry, Q is the slurry flow rate and im is the average (global) beach slope.

TABLE I. AVERAGE PHYSICAL CHARACTERISTICS OF THE TAILINGS USED IN THE HYDRAULIC DEPOSITION SIMULATION TEST HDST [3].

	Quartz	**Iron**
D50 (mm)	0.265	0.240
D90 (mm)	0.645	0.640
Gs	2.65	5.50

TABLE II. CONTROLLED VARIABLES AND FINAL SLOPE INCLINATION OF THE HYDRAULICALLY DEPOSITED BEACH [3].

	HDST 1	**HDST 6**
Cw (%)	8.9	20.4
Fe (%)	23.0	23.0
Q (l/min)	4.8	5.9
im (%)	7.7	9.2

Figures 6 and 7 present comparisons between the results obtained from tests HDST 1 and 6 by [3] and those obtained analytically using the mathematical model proposed by [4], and numerically using the FDM as obtained by [2] and using the CIP technique describe in [5] for a Courant number equal to 0.5. Axis x (abscissas) represents the distance from the discharge point and axis y (ordinates) gives the normalized height of the deposited beach.

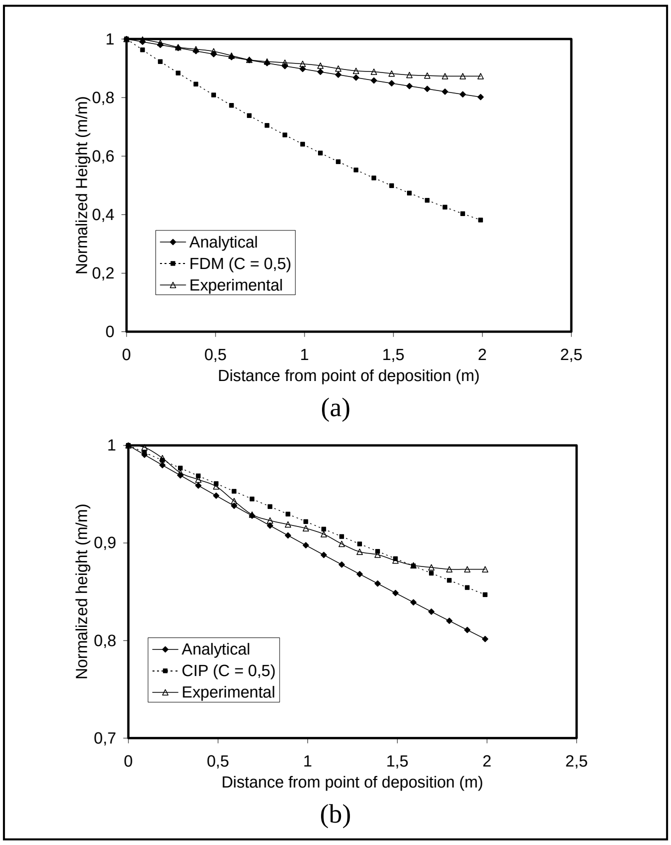

Figure 6. Comparisons between experimental data for test HDST 1 (Ribeiro [3]) and numerical results using: (a) Finite Differences; and (b) CIP method for Courant number C=0.5.

From Figures 6 and 7, one can notice that the mathematical model is not able to describe the successive erosion and deposition processes, which are clearly observed in the HDST beaches. However, the model describes quite well the basic geometric characteristics of the deposited beaches, such as global slope inclination. The numerical solution using the Finite Differences Method differs drastically from the experimental and analytical solution for the adopted time-space discretization with C=0.5. This is due to spurious dissipation introduced by the numerical solution scheme as explained in [5]. This problem is solved when the mixed CIP-FDM scheme is adopted and the numerical solution matches satisfactorily the experimental and analytical results for any Courant number, as illustrated in figures 6(b) and 7(b) for C=0.5

VI. CONCLUSIONS

A coupled system of equations representing the bed load transport of heterogeneous sediments in hydraulically deposited tailings dams was described. This model is governed by the field equations for equilibrium and continuity of the fluids and the condition of continuity of the solid particles. A constitutive model must also be assumed for the rate of transport of the sediments.

The overall system of equations governing the problem may be split in a sub-system of advective equations and a sub-system of non advective equations. Advective equations do not exhibit dissipation during the transport phenomena; however spurious numerical dissipations are introduced in the solution when classical schemes, such as the Finite Differences Method (FDM), are used to solve this kind of hyperbolic equations.

The solution for the unrealistic numerical dissipation problem may be achieved by adopting the so-called CIP (Cubic Interpolated Pseudo-particle) method to solve the advective part of the equations governing the overall problem. The remaining equations may be solved using any other method, such as the FDM, without undesirable numerical problems.

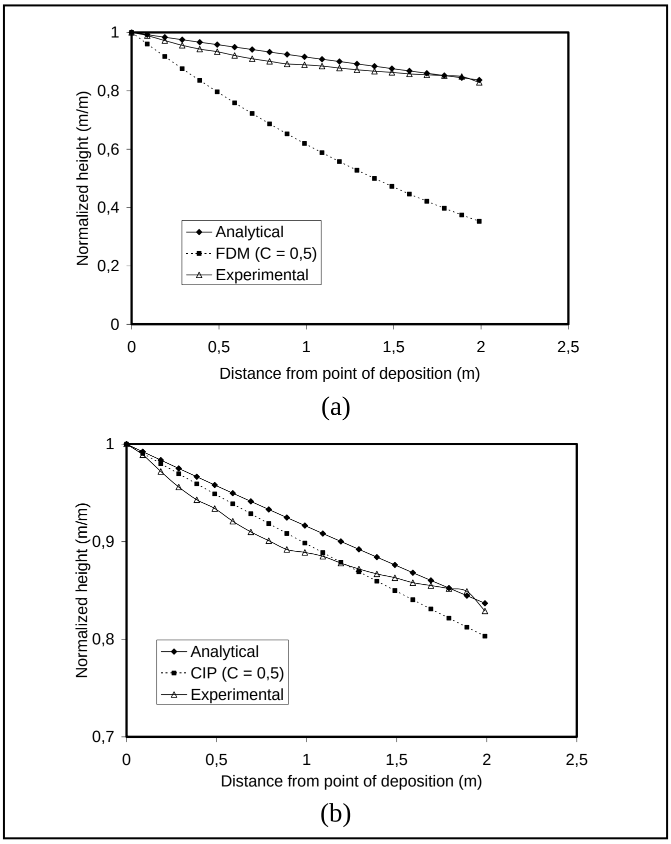

Figure 7. Comparisons between experimental data for test HDST 6 (Ribeiro [3]) and numerical results using: (a) Finite Differences; and (b) CIP method for Courant number C=0.5.

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