International Journal of
Science and Engineering Investigations vol. 1, issue 6, July 2012
ISSN: 2251-8843

A Fast Algorithm for Computing Pseudospectra of Companion Matrices

Zhengsheng Wang[1], Chuntao Liu[2]
[1, 2]Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, China
([1]wangzhengsheng@nuaa.edu.cn, [2]liuchuntao9@126.com)

Abstract-The concept of pseudospectra was introduced by Trefethen during the 1990s and became a popular tool to explain the behavior of non-normal matrices. It is well known that the zeros of a polynomial are equal to the eigenvalues of the associated companion matrix. It is feasible to do the sensitivity analysis of the zeros of polynomials by the tool of pseudospectra of companion matrices. Thus, the pseudospectra problem of companion matrices is meaningful. In this paper, we propose a fast algorithm for computing the pseudospectra of companion matrices by using fast QR factorization. At last, numerical experiments and comparisons are given to illustrate(confirm) the efficiency of the new algorithm.

Keywords*-Pseudospectra; Companion matrix; Grid; SVD; QR factorization; Field of values; Gershgorin disk*

I. INTRODUCTION

The concept of pseudospectra was introduced by Trefethen to explain the behavior of non-normal matrices [1, 2, 3, 4]. The pseudospectra of a square matrix are the set of all eigenvalues of complex matrices within a given distance. It is a useful tool for understanding the behavior of various matrix processes. Many phenomena (for example, hydrodynamic instability and convergence of iterative methods for linear systems) can not be accounted for by eigenvalue analysis but are more understandable by examining the pseudospectra [5, 6, 7, 8].

There are the following equivalent definitions of pseudospectra. Let $\|\cdot\|$ be a matrix norm induced by a vector norm.

(1) $\Lambda_\varepsilon(A)=\left\{z\in C:\left\|(zI-A)^{-1}\right\|\geq\varepsilon^{-1}\right\}$;

(2) $\Lambda_\varepsilon(A)=\left\{z\in C: z\in(A+E),\text{for some } E \text{ with } \|E\|\leq\varepsilon\right\}$

(3) $\Lambda_\varepsilon(A)=\left\{z\in C:\text{there exists } v\in C^N,\|v\|=1,\text{such that } \|(A\text{-}zI)v\|\leq\varepsilon\right\}$

(4) If $\|\cdot\|$ is the the 2-norm, the following definition is also equivalent:

$\Lambda_\varepsilon(A)=\left\{z\in C:\sigma_{\min}(zI\text{-}A)\leq\varepsilon\right\}$;

Note that if $zI\text{-}A$ is singular, we denote $\left\|(A\text{-}zI)^{-1}\right\|=\infty$ and $\sigma_{\min}(\cdot)$ denotes the minimum singular value. From these definitions, it immediately follows that for any $\varepsilon_1>\varepsilon_2$,

$\Lambda_{\varepsilon_1}(A)\supseteq\Lambda_{\varepsilon_2}(A)$.Furthermore, $\Lambda_0(A)$ coincides with the eigenvalues of A.Especially, if matrix A is normal, the $\Lambda_\varepsilon(A)$ is just the closed " ε -neighborhood of $\Lambda(A)$.

The popular computational method for matrix pseudospectra are the grid method based on basic SVD (denoted Grid-SVD algorithm). Also, there are other computational methods, such as inverse iteration, Lanczos iteration, Arnoldi iteration, continuation and so on [2, 3, 4, 5, 6]. And, pseudospectra of rectangular matrices have been considered by Wright and Trefethen [11].

Algorithm 1.1 The Grid-SVD algorithm

1. Construct a mesh Ω over a region of the complex plane that includes $\Lambda_\varepsilon(A)$;
2. Compute $\sigma_{\min}(zI-A)$ for every node z of Ω;
3. Contour the pseudospectra figure.

On the other hand, as we know, for each monic polynomial

$$p(z)=z^n+c_{n-1}z^{n-1}+\cdots+c_1z+c_0 \tag{1}$$

the companion matrix associated with $p(z)$ is the $n\times n$ matrix

$$A_p=\begin{bmatrix}0 & \cdots & \cdots & -c_0\\ 1 & \ddots & & -c_1\\ & \ddots & \ddots & \vdots\\ & & 1 & -c_{n-1}\end{bmatrix} \tag{2}$$

The characteristic polynomial of A_p is $p(z)$ itself, and therefore the set of eigenvalues of A_p coincides with the set of zeros of $p(z)$.In another word, the zeros of a

polynomial $p(z)$ are equal to the eigenvalues of the associated companion matrix A.

In this paper, we propose a fast algorithm for computing the pseudospectra of companion matrices by using fast QR factorization in section 2. Numerical experiments and comparisons are given to illustrate the efficiency of the new algorithm in section 3. Conclusion and remarks are given in section 4.

II. A FAST ALGORITHM FOR COMPUTING PSEUDOSPECTRA OF COMPANION MATRIX

An equivalent definition of matrix pseudospectra was introduced in [8, 9] as follows.Suppose a matrix written as follows

$$A=[\alpha_1,\alpha_2,\cdots,\alpha_n]$$

It is shown that a system of vectors $\{\alpha_1,\alpha_2,\cdots,\alpha_n\}$ is dependence if and only if $G[\alpha_1,\alpha_2,\cdots,\alpha_n]=0$, where $G[\alpha_1,\alpha_2,\cdots,\alpha_n]$ is Gram determinant , $G[\alpha_1,\alpha_2,\cdots,\alpha_n]=\det(A^T A)$. The vector α_{r+1} about $Span\{\alpha_1,\alpha_2,\cdots,\alpha_n\}$ relative dependence index is defined as follows

$$E\left[\alpha_{r+1}\middle|\alpha_1,\alpha_2,\cdots,\alpha_n\right]=\min_{c_i\in C}\left\|\alpha_{r+1}-\sum_{i=1}^{r}c_i\alpha_i\right\|_2 \quad (3)$$

Geometric interpretation of (2) is that $E\left[\alpha_{r+1}\middle|\alpha_1,\alpha_2,\cdots,\alpha_r\right]$ is the length of the orthogonal projection of α_{r+1} onto the space $Span\{\alpha_1,\alpha_2,\cdots,\alpha_r\}$ .According to geometric property of Gram determinant, we have the following equality

$$E\left[\alpha_{r+1}\middle|\alpha_1,\alpha_2,\cdots,\alpha_r\right]=\sqrt{\frac{G[\alpha_1,\alpha_2,\cdots,\alpha_{r+1}]}{G[\alpha_1,\alpha_2,\cdots,\alpha_r]}} \quad (4)$$

Based on a theory of numerical linear dependence we get a new definition of pseudospectra. For any given $z\in C$ and each $\varepsilon\geq 0$, let $B=zI-A=[\beta_1,\beta_2,\cdots,\beta_n]$. The pseudospectra of matrix A can be defined by

$$\bar{\Lambda}_\varepsilon(A)=\left\{z\in C\middle|E\left[\beta_n\middle|\beta_1,\beta_2,\cdots,\beta_{n-1}\right]\leq\varepsilon\right\} \quad (5)$$

Before we present a new algorithm for computing pseudospectra of companion matrix, we need to introduce the fast QR factorization of upper-Hessenberg matrix. The cost of the fast QR factorization of upper-Hessenberg matrix using Givens rotations is $O(n^2)$ operations because of its upper-Hessenberg structure while it is $O(n^3)$ operations for QR factorization of general matrix. And, the cost of the singular value decomposition (SVD) of general matrix is also $O(n^3)$ operations. Therefore, we can present a fast algorithm for computing pseudospectra of companion matrix which needs much less cost and is much faster and than the traditional algorithm (Grid SVD).

Because a companion is an upper-Hessenberg matrix, we consider the QR factorization of upper Hessenberg matrices using Givens rotations. In detail, given a 2×2 matrix

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

the rotation matrix

$$Q_{12}=\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

With $\tan\theta=\dfrac{a_{21}}{a_{11}}$, zeros the (2; 1) position of the matrix A.

For an $n\times n$ upper-Hessenberg matrix A,let

$$Q^H=Q_{n-1,n}Q_{n-2,n-1}\cdots Q_{2,3}Q_{1,2}$$

Then $Q^H A=R$, where R an upper-triangular matrix. This generates the QR factorization of A.

Suppose the matrix $B=zI-A_p=[\beta_1,\beta_2,\cdots,\beta_n]$ has the following fast QR factorization using Givens rotations.

$$B=zI-A_p=[\beta_1,\beta_2,\cdots,\beta_n]$$
$$=Q\begin{bmatrix} r_{11} & * & \cdots & * \\ & r_{22} & \ddots & \vdots \\ & & \ddots & * \\ & & & r_{nn} \end{bmatrix}=QR \quad (6)$$

With $|r_{11}|\geq|r_{22}|\geq\cdots\geq|r_{nn}|$ and Q is an $n\times n$ unitary matrix.

If z is one of eigenvalues of A_p, $B=zI-A_p$ is singular; otherwise, $|r_{nn}|\neq 0$.

And

$$G[\beta_1,\beta_2,\cdots,\beta_n]=\det(B_k^T B)=\det(R_k^T R)=r_{11}^2 r_{22}^2\cdots r_{kk}^2$$

Then we get

$$E\left[\beta_n \,|\, \beta_n,\beta_2,\cdots,\beta_{n\text{-}1}\right]=|r_{nn}|$$

From the definition of the minimum singular value of the matrix B,

$$\sigma_{\min}\left(zI-A_p\right)=\sigma_{\min}\left(B\right)=\min_{\|x\|_2=1}\|Bx\|_2$$

Since $B=QR$ and the unitary invariance of the 2-norm, let $x=e_n$ we have

$$\sigma_{\min}\left(zI-A_p\right)\le|r_{nn}|$$

This formula implies

$$\bar{\Lambda}_\varepsilon\left(A_p\right)\subseteq\Lambda_\varepsilon\left(A_p\right)$$

Since the singular values and r_{nn} are continuous functions of the matrix entries,hence if $r_{nn}\to 0$ then $\sigma_{\min}\left(B\right)\to 0$ .The converse is also true. It also can be seen that $\bar{\Lambda}_\varepsilon\left(A_p\right)$ and $\Lambda_\varepsilon\left(A_p\right)$ change continuously with $\varepsilon>0$ .If a system of vectors $\{\beta_1,\beta_2,\cdots,\beta_n\}$ is orthogonal system then we have $\sigma_{\min}\left(B\right)=|r_{nn}|$, since $B^TB=diag\left[r_{11}^2,r_{22}^2,\cdots,r_{nn}^2\right]$. This suggests that under assumption of orthogonal system the definition of pseudospectra is equivalent to Trefethen's definition of pseudospectra. In this case $\bar{\Lambda}_\varepsilon\left(A_p\right)=\Lambda_\varepsilon\left(A_p\right)$.

Then we propose a fast algorithm for computing pseudospectra of matrices described as follows.

Algorithm 2.1 The Grid-QR algorithm

1. Construct a mesh Ω over a region of the complex plane that includes $\Lambda_z\left(A\right)$;
2. Compute $B=zI-A_p=\left[\beta_1,\beta_2,\cdots,\beta_n\right]$ for every node z of Ω;
3. Compute fast QR factorization of matrix B using Givens rotations;
4. If $|r_{nn}|/|r_{11}|\le\varepsilon$, then $z\in\bar{\Lambda}_\varepsilon\left(A\right)$, else go to step 1;
5. Contour the pseudospectra figure.

III. NUMERICAL EXPERIMENTS

In this section, we give some numerical experiments and comparisons to confirm the results in this paper. Here, all the computations are finished with MATLAB 6.5 on PC (Intel(R) Pentium(R), Processor 1500MHz 1.50GHz, Memory 256MB).

Example 1 Consider the $n\times n$ companion matrix associated with the polynomial whose roots are 1,2,· · ·,n. In other word, consider the monic polynomial $p\left(z\right)$ of degree n with zeros 1,2,…,n. The pseudospectra of the associated companion matrices with $n=5,10,20$ are plotted in Figure 1-Figure3.

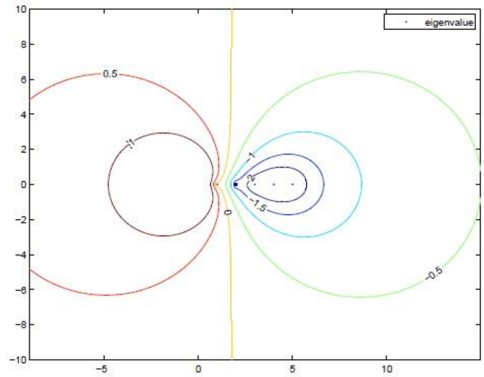

Figure 1: pseudospectra of the 5×5 companion matrix with $\varepsilon=10^p\left(p=-2:0.5:1\right)$ on 200×200 grids on the region $\left[-9,15,-10,10\right]$

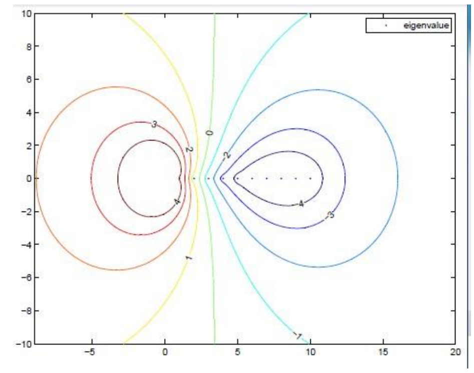

Figure 2: pseudospectra of the 10×10 companion matrix with $\varepsilon=10^p\left(p=-4:1:4\right)$ on 200×200 grids on the region $\left[-9,15,-10,10\right]$

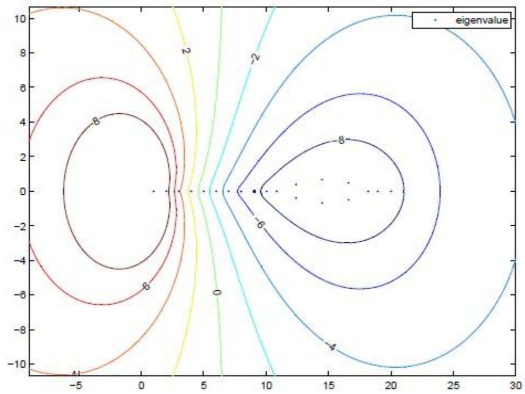

Figure 3: pseudospectra of the 20×20 companion matrix with $\varepsilon=10^p\left(p=-8:2:8\right)$ on 200×200 grids on the region $\left[-9,15,-10,10\right]$

Table I shows the comparisons between different size n with the same number of the grid points $\left(200\times 200\right)$.

www.IJSEI.com ISSN: 2251-8843 Paper ID: 10612-08

TABLE I. THE CPUTIME COMPARISON

n	Cond (A)	Grid-SVD CPU time	Grid-QR CPU time
5	1.2296E+003	0.7685	0.3242
10	1.0252E+008	1.5445	0.6910
20	Infinite	4.4764	1.9969

Example 2 Consider the polynomial $p(z)=z^n+2z^{n-1}+3z^{n-2}+4z^{n-3}+\cdots(n-1)z^2+nz+(n+1)$ The pseudospectra of the companion matrices with n=5,10 are plotted in Figure 4-Figure 5.

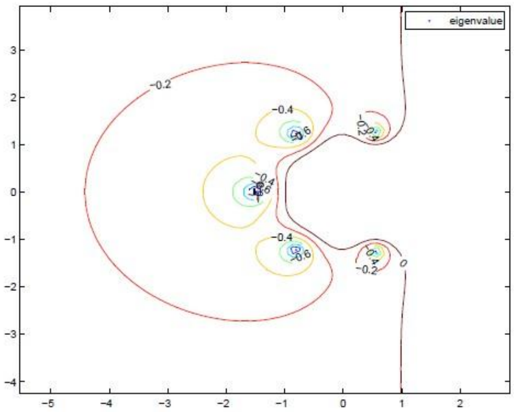

Figure 4: pseudospectra of the 5×5 companion matrix with $\varepsilon=10^p\ (p=-2{:}0.2{:}0)$ on 200×200 grids.

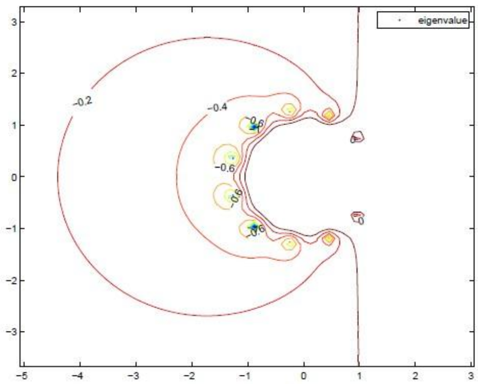

Figure5: pseudospectra of the 10×10 companion matrix with $\varepsilon=10^p\ (p=-2{:}0.2{:}0)$ on 200×200 grids.

IV. CONCLUDING REMARKS

Perhaps pseudospectra will play a role tool in breaking down walls between the theorists of functional analysis and the engineers of scientific computing. Computing pseudospectra will be a routine matter among scientists and engineers who deal with non-normal matrices [1]. Thus, we do the interesting research work on the pseudospectra of matrices.

In this paper, we propose a fast efficient algorithm for computing the pseudospectra of companion. And we discuss the approach for computing the pseudospectra region easily. Furthermore, we discuss the relationship between pseudozeros sets of polynomial and pseudospectra of the associated companion matrix. The numerical experiments and comparisons are given to confirm the results in this paper.

In fact, the idea in this paper can be generalized to the pseudospectra of rectangular matrices and other structure matrices. Furthermore, the idea in this paper may provide some insights for the pesudospectra in other norms, i.e. weighted pseudospectra.

ACKNOWLEDGMENT

This work was supported by the Fundamental Research Funds for the Central Universities (No.NZ2012307).

REFERENCES

[1] L.N.Trefethen. Spectra and pseudospectra: The behavior of nonnormal matrices and operator. Princeton University Press, 2005, Princeton.

[2] L.N.Trefethen. Pseudospectra of linear operators. SIAM Review, 1997,39(3):383-406.

[3] L.N.Trefethen. Computation of pseudospectra. Acta Numerica ,1999, Cambridge:Cambridge University Press, 247-295.

[4] T.G.Wright,L.N.Trefethen. Large-scale computation of pseudospectra using ARPACK and eigs, SIAM J.Sci. Comput., 2001, 23(2):591-605.

[5] K.C.Toh, and L.N.Trefethen, Calculation of pseudospectra by the Arnoldi iteration, SIAM J.Sci. Comput., 1996, 17(1):1-15.

[6] T.Braconnier, and N.J.Higham, Computing the field of values and pseudospectra using the Lanczos method with continuation, BIT, 1996,36(3):422-440.

[7] M.Bruhl. A curve tracing algorthim for computing the pseudospectrum.BIT, 1996, 36(3):441-454.

[8] Y. Shen, J. Zhao, H. Fan. Properties and computations of matrix pseudospectra.Applied mathematics and Computation, 2005, 161: 385-393.

Zhengsheng Wang. Born in 1972, he received the Bachelor degree, Master degree and Ph.D. of Computational Mathematics in Engineering in Nanjing University of Aeronautics and Astronautics in China. He has a academic access Swinburne University of Technology in Australia.

He is an associate professor of Computational Mathematics in Nanjing University of Aeronautics and Astronautics. Since 2000, he has involved in the completion of five projects with national, provincial or departmental level. And now he has three projects for researching. He has published over 20 academic papers including 8 for SCI, EI or ISTP retrieval.

Chuntao Liu. Born in 1987, she received the Bachelor of Mathematics in Fuyang Normal College in China; and now studying in Nanjing University of Aeronautics and Astronautics as a graduate of Computational Mathematics.