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Solving Linear Systems Using Interval Arithmetic Approach

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Abstract: in this paper we discuss various classes of solution sets for linear interval systems of equations, and interval linear programming problems. And their properties, in this model we let the coefficient matrix and the right vector hands and the cost coefficient are interval. Interval methods constitute an important mathematical and computational tool for modeling real-world systems (especially mechanical) with bounded uncertainties of parameters, and for controlling rounding errors in computations. They are in principle much simpler than general probabilistic or fuzzy set formulation, while in the same time they conform very well to many practical situations. Linear interval systems constitute an important subclass of such interval models, still in the process of continuous development.

Keywords- *linear interval systems; interval arithmetic; interval model*

I. INTRODUCTION

In many real-life problems we deal with a mathematical programming problem. In conventional mathematical programming coefficient of problem are usually determined by the experts as crisp values. But in reality in an imprecise and uncertain environment, it is an unrealistic assumption that the knowledge and representation at an expert are so precise. Then we know, at best, the intervals of possible values. Thus it is desirable to analyze how the corresponding mathematical results will look if we replace numbers by intervals.

Let $a=\left[\overline{a},\underline{a}\right]=\left\{x:\overline{a}\le x\le\overline{a}, x\in R\right\}$,

Where $\overline{a}$ and $\underline{a}$ are the left and right limits of the interval a on the real line R, respectively. We shall use the terms "interval" and "interval number" interchangeably. If $\underline{a}=\overline{a}=a$, then $a=\left[\underline{a},\overline{a}\right]$ is a real number. We use IR to denote the set of all interval numbers on the real line R. Interval arithmetic was first suggested by Dwyer [1] in 1951. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore [2]. After this motivation and inspiration, several authors such as (Alefeld and Herzberger [3], Dubois et al [4], Hansen [5],

In order to develop good mathematical programming methodology interval approaches are frequently used to describe and treat imprecise and uncertain elements present in a real decision problem. Linear systems of equations are among the most frequently used tools in applied mathematics. The solution to linear systems of equations is prone to errors due to the finite precision of machine arithmetic and the propagation of error in the initial data. If the initial data is known to lie in specified ranges then interval arithmetic enables computation of intervals containing the elements of the exact solution.

Solution of linear interval system of algebraic equations is a challenging problem in interval analysis and robust linear algebra. This problem was first considered at the middle of 1960s by Oettli and Prager[6] and was pointed out as very important for numerous applications. Since that, this problem has received much attention and was developed in the context of modeling of uncertain systems (see [7]).

Consider a system of linear algebraic equations

$$Ax=b \qquad (1)$$

with $x\in R_n$, interval matrix $A\in IR_{n\times n}$ and interval vector $b\in IR_n$. The matrix and vector are said to belong to interval family if their elements are from some real intervals [a; b], $a\le b$. Here the standard notations $IRn\times n$ and IRn are used for sets of all n-dimensional interval square matrices and vectors, respectively. System (1) is called the interval system of equations.

II. INTERVAL ARITHMETIC

The main principle of interval arithmetic is to replace every real number by an interval enclosing it and whose bounds are representable by the computer [5] For instance, π can be represented by the interval [3.14159, 3.14160] if 6 significant radix-10 digits are used Data known with some degree of uncertainty can also be represented, for instance data measured with bounded measurement errors. Interval vectors and interval matrices are vectors and matrices with interval components .The major advantage of this arithmetic is the fact that every result is guaranteed.

III. BASICS TOOLS

An interval $[X]=[\underline{x},\overline{x}]$ is a closed and connected subset of R ; it may be characterized by its lower and upper bounds $\underline{x}$ and $\overline{x}$ or equivalently by its center

$$c([x])=\frac{(\underline{x}+\overline{x})}{2}=m(x)$$

and width $w([x])=\overline{x}-\underline{x}$ .Arithmetical operations on intervals can be defined by

$$\forall\circ\in\{+,-,*,/\},[x]\circ[y]=\{x\circ y|x\in[x],y\in[y]\}$$

Obtaining an interval corresponding to $[x]\circ[y]$ is easy for the first three operators as

$$[x]+[y]=[\underline{x}+\underline{y},\overline{x}+\overline{y}],\ [x]-[y]=[\underline{x}-\overline{y},\overline{x}-\underline{y}],$$

$$[x]*[y]=[\min(\overline{x}\overline{y},\overline{x}\underline{y},\underline{x}\overline{y},\underline{x}\underline{y}),\max(\overline{x}\overline{y},\overline{x}\underline{y},\underline{x}\overline{y},\underline{x}\underline{y})]$$

For division, when $0\notin[y]$,

$[x]/[y]=[\min(\overline{x}/\overline{y},\overline{x}/\underline{y},\underline{x}/\overline{y},\underline{x}/\underline{y}),\max(\overline{x}/\overline{y},\overline{x}/\underline{y},\underline{x}/\overline{y},\underline{x}/\underline{y})]$ and *extended intervals* have to be introduced when $0\in[y]$, see[6].

More generally, the *interval counterpart* of a real-valued function is an interval-valued function defined as $f([x])=[\{f(x)|x\in[x]\}]$,

Where $[S]$ is the *interval hull* of S , *i.e.*, the smallest interval that contains it. Intervals to continuous elementary functions are easily obtained. For monotonic functions only computations on bounds are required.

$$\exp([x])=[\exp(\underline{x}),\exp(\underline{x})],$$

$$\log([x])=[\log(\underline{x}),\log(\underline{x})]\ \text{if}\ \underline{x}>0$$

For non-monotonic elementary functions, such as the trigonometric functions, algorithmic definitions are still easily obtained. For instance, the interval square function can be defined by

$$[x]^2=\begin{cases}\left[0,\max\left(\underline{x}^2,\overline{x}^2\right)\right], if\ 0\in[x]\\ \left[\min\left(\underline{x}^2,\overline{x}^2\right),\max\left(\underline{x}^2,\overline{x}^2\right)\right] else\end{cases}$$

For more complicated functions, it is usually no longer possible to evaluate their interval counterpart, hence the importance of the concept of *inclusion function*. An inclusion function $[f](.)$ for a function $f(.)$ defined over a domain $D\subset R$ is such that the image of an interval by this function is an interval, guaranteed to contain the image of the same interval by the original function:

$$\forall[x]\subset D,\ f([x])\subset[f]([x])\qquad(2).$$

This inclusion function is *convergent* if $\lim_{w([x])\to 0} w([f]([x]))=0$ and *inclusion Monotonic if* $[x]\subset[y]\Rightarrow[f]([x])\subset[f]([y])$.

Various techniques are available for building convergent and inclusion-monotonic inclusion functions. Among them, the simplest is to replace all occurrences of the real variable by its interval counterpart which results in what is called a natural inclusion function.

Example 3.1 Consider the function $f(x)=x^2-3(x-\exp(x))$.

An inclusion function for f is

$$[f]([x])=[x]^2-3([x]-\exp([x]))$$

Evaluate $[f]$ *over* $[0,1]$,

$$[f]([0,1])=[0,1]^2-3([0,1]-\exp([0,1]))$$

When

$$=[0,1]-3([0,1]-[1,e])=[0,1]-3([-e,0])$$

$$=[0,1]+[0,3e+1]=[0,3e+1]\subset[0,9.16]$$

compare with

$$f([0,1])=[3,-2+3e]\subset[3,6.16]$$

of course, $f([0,1])\subset\ [f]([0,1])$

the inclusion in (2) becomes an equality, the inclusion function is *minimal*. Usually, some pessimism is introduced by the inclusion function, as in example3.1

This pessimism is due to the fact that each occurrence of the interval variable is considered as independent from the others. Various approaches may be considered to reduce pessimism. A first one is to reduce the number of occurrences of the variable by symbolic manipulations.

IV. LINEAR SYSTEMS OF EQUATIONS

The detailed description of the solution set for linear interval systems was given in the pioneer work by Oettli and Prager [6] for general situation of interval uncertainty, their result is reduced as follows.

We assume that A is nonsingular, in which case the solution x exists and is unique. Now, suppose that the elements in A and b are uncertain, but we know bounds for each of them. We can use these bounds as end points of intervals, and replace (1) with

$$AX = b \qquad (1.1)$$

Where A and b are the interval matrix and vector as described above, and we $X \in IR^n$ that contains the solution to every problem en-compassed in (1.1). We assume that A is regular, meaning that every real matrix contained in A is nonsingular.

EXAMPLE 4.1

$$[2,3]x_1 + [-1,2]x_2 = [3,4]$$
$$[1,3]x_1 + [4,6]x_2 = [2,4]$$

A vector $x = (x_1, x_2) \in X$ must satisfy both equations. The first equation can be reformulated to

$$[2,3]x_1 = [3,4] - [-1,2]x_2,$$

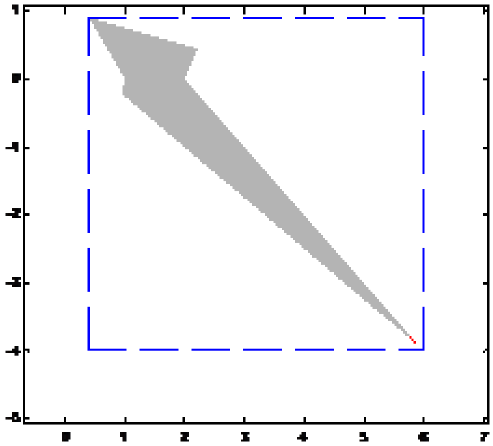
which has the solution

$$x_1 = \begin{cases} \left[\frac{3}{2} - x_2, 2 + \frac{1}{2}x_2\right] & \text{for } x_2 \ge \frac{3}{2} \\ \left[1 - \frac{2}{3}x_2, 2 + \frac{1}{2}x_2\right] & \text{for } 0 \le x_2 \langle \frac{3}{2} \\ \left[1 + \frac{1}{3}x_2, 2 - x_2\right] & \text{for } - \le x_2 \langle 0 \\ \left[\frac{3}{2} + \frac{1}{2}x_2, 2 - x_2\right] & \text{for } x_2 \langle -3 \end{cases}$$

Similarly, we find that the points satisfying the second equation are given by

$$x_2 = \begin{cases} \left[\frac{1}{2} - \frac{3}{4}x_1, \frac{2}{3} - \frac{1}{6}x_1\right] & \text{for } x_1 \ge 4 \\ \left[\frac{1}{2} - \frac{3}{4}x_1, 1 - \frac{1}{4}x_1\right] & \text{for } \frac{2}{3} \le x_1 \le 4 \\ \left[\frac{1}{2} - \frac{3}{4}x_1, 1 - \frac{1}{4}x_1\right] & \text{for } \frac{2}{3} \le x_1 \le 4 \\ \left[\frac{1}{2} - \frac{3}{4}x_1, 1 - \frac{1}{4}x_1\right] & \text{for } \frac{2}{3} \le x_1 \le 4 \end{cases}$$

The set of solutions to the system is the intersection of the two domains illustrated in the figureIV.1



FigureIV.1. the set of solutions

we cannot give such a figure as the result, but have to make do with the interval hull, marked by the discontinued rectangle in the figure. Thus, the system has the solution

$$X = \begin{pmatrix} [0.4, 6] \\ [-4, 0, 9] \end{pmatrix}$$

A. GAUSSIAN ELIMINATION

An obvious approach is to use a generalization of Gaussian elimination adapted to deal with interval coefficients. A triangular system can be formed in the usual way but with interval arithmetic. By the inclusion property, the solution of this triangular system will give an inclusion of the solution set.

The usual care has to be taken with division by zero. Column magnitude pivoting can be used to choose a pivot as the contender with the largest magnitude, where we recall that the magnitude of x is defined as

An implementation written in INTLAB [13] of interval Gaussian elimination with magnitude pivoting is given by the function intgauss.m[8] .

When interval Gaussian elimination is applied to a general $A \in IR^{n\times n}$ and $b \in IR^n$ problems are soon encountered as n is increased.

The vector interval obtained by applying a Gaussian elimination algorithm to the system defined in the exampleIV.1 is

$$X = \begin{pmatrix} [-1.5, 6] \\ [-4, 3.0001] \end{pmatrix}$$

As interval calculations are carried out in the Gaussian elimination process the widths of the interval components grow larger due to the nature of interval arithmetic. If a solution is obtained then it is likely that the width of the components is very large. Alternatively, at some stage in the Gaussian elimination process all contenders for pivot, or the bottom right element in the upper triangular system, contain zero, which causes the algorithm to break down due to division by zero [9].

The feasibility of using intgauss.m depends on the matrix $A \in IR^{n\times n}$. For a general A, problems may occur for dimensions as low as n = 3 if the radii of the elements are too large. As the width of the elements decreases the algorithm becomes feasible for larger n. However, even when intgauss.m is used with thin matrices, it is likely for the algorithm to break down for n larger than 60.

Despite interval Gaussian elimination not being effective in general, it is suitable for certain classes of matrices. In particular, realistic bounds for the solution set are obtained, for M-matrices, H-matrices, diagonally dominant matrices, tridiagonal matrices. In the case where A is an M-matrix the exact interval hull is obtained for many b; Neumaier [7] shows that if $b \geq 0, b \leq 0$ *or* $0 \in b$ then the interval hull of the solution set is obtained.

B. KRAWCZYK'S METHOD

The linear interval system $AX = b$ can be preconditioned by multiplying by a matrix. $C \in R^{n\times n}$. Here, we choose C to be the inverse of the midpoint matrix of A, which often leads to the matrix CA being an H-matrix. If this is the case then Gaussian elimination can be used, but it is quicker to compute an enclosure of the solution by Krawczyk's method.

Assuming an interval vector $x^{(i)}$ is known such that $\square\Sigma\left(A,b\right) \subseteq x^{(i)}$ then

$$\tilde{A}^{-1}\tilde{b} = C\tilde{b} + (I - C\tilde{A})\tilde{A}^{-1}\tilde{b} \in Cb + \left(I - CA\right)x^{(i)}$$

Holds for all $\tilde{A} \in A$ *and* $\tilde{b} \in b$ *so that*

$\square\Sigma\left(A,b\right) \subseteq x^{(i)} \Rightarrow \square\Sigma\left(A,b\right) \subseteq (Cb + \left(I - CA\right)x^{(i)}) \cap x^{(i)}$ This gives the Krawczyk iteration

$$x^{(i+1)} = (Cb + \left(I - CA\right)x^{(i)}) \cap x^{(i)}$$

To start the iteration we require an initial vector $x^{(0)}$ such that the solution $x^{(0)} \supseteq \square\Sigma\left(A,b\right)$

C. Determinant method

Theorem*:* Let $Ax = b$ be a system of linear equations involving interval numbers. If the (n × n) interval matrix A is invertible, then it is possible to find a smallest box $\mathbf{x} = (x_1, x_2, x_3, ..., x_n)$ which containing the exact solution of the system (1). Where each

$$x_i = \frac{\left|A^{(i)}\right|}{\left|A\right|}$$

$A^{(i)}$ is the interval matrix obtained when the it column of A is replaced by the vector $b = (b_1, b_2, ..., b_n)$, $\left|A^{(i)}\right|$ and $\left|A\right|$ are the ad joint matrix and the determinant matrix of A respectively .

Example 5: We consider an example given in Ning et al [12].

The system of interval equations $AX = b$ be given

$$\text{with} \quad A = \begin{pmatrix} [3.7, 4.3] & [-1.5, -0.5] & [0, 0] \\ [-1.5, -0.5] & [3.7, 4.3] & [-1.5, -0.5] \\ [0, 0] & [-1.5, -0.5] & [3.7, 4.3] \end{pmatrix} \text{and}$$

$$b = \begin{pmatrix} [-14, 0] \\ [-9, 0] \\ [-3, 0] \end{pmatrix}.$$

Here $|A| = \left[37.103, 74.89\right]$ *and* $|A| \neq 0$

$$\text{Now} \; |A^{(1)}| = \begin{pmatrix} [-14, 0] & [-1.5, -0.5] & [0, 0] \\ [-9, 0] & [3.7, 4.3] & [-1.5, -0.5] \\ [-3, 0] & [-1.5, -0.5] & [3.7, 4.3] \end{pmatrix}$$

$$= \left[-249, 0\right],$$

$$|A^{(2)}| = \begin{pmatrix} [3.7, 4.3] & [-14, 0] & [0, 0] \\ [-1.5, -0.5] & [-9, 0] & [-1.5, -0.5] \\ [0, 0] & [-3, 0] & [3.7, 4.3] \end{pmatrix}$$

$$= \left[-212, 0\right] \text{and}$$

$$|A^{(3)}| = \begin{pmatrix} [3.7, 4.3] & [-1.5, -0.5] & [-14, 0] \\ [-1.5, -0.5] & [3.7, 4.3] & [-9, 0] \\ [0, 0] & [-1.5, -0.5] & [-3, 0] \end{pmatrix}$$

$$= \left[-98, 3\right].$$

Then by the above theorem we see that

$$x_1 = \frac{\left[-249, 0\right]}{\left[37.103, 74.89\right]} = \left[-4, 482, 0\right],$$

$$x_1 = \frac{\left[-212, 0\right]}{\left[37.103, 74.89\right]} = \left[-3.816, 0\right] \text{and}$$

$$x_1 = \frac{[-98, 0]}{[37.103, 74.89]} = [-1.776, 0.006].$$

In this case, the solution set is

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} [-4,482, 0] \\ [-3.816, 0] \\ [-1.776, 0.006] \end{pmatrix}.$$

Using interval Gaussian elimination with interval arithmetic.

$$x = \begin{pmatrix} [\ -5.7081, & 0.6415] \\ [\ -4.7466, & 0.0000] \\ [\ -2.7351, & 0.0000] \end{pmatrix}.$$

Using interval hull method with interval arithmetic we obtained the solution set (much wider)

$$x = \begin{pmatrix} [\ -5.7770, & 1.4437] \\ [\ -4.8173, & 1.4840] \\ [\ -2.7921, & 1.2088] \end{pmatrix}$$

It is to be noted that the solution set obtained by using interval approach developed by determinant method is sharper then the solution sets obtained by other techniques.

V. CONCLUSION

Intervals containing the elements of the exact solution.

The solution set obtained by using interval approach developed by determinant method is sharper then the solution sets obtained by other techniques using interval arithmetic.

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